Symmetry Breaking in Static and Dynamic Networks

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Rate of Updates in Networks

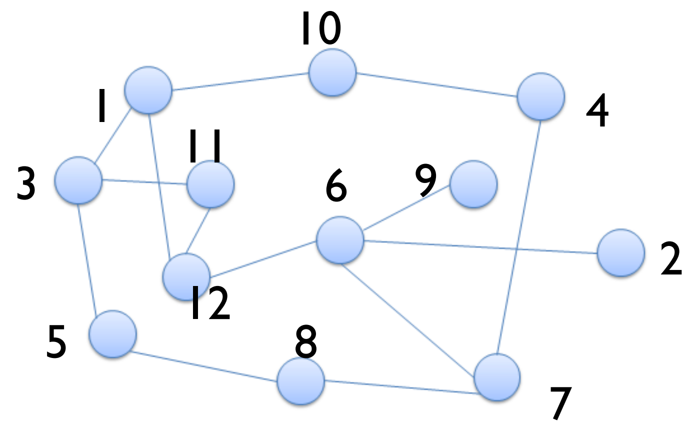
(Estimation)

Vertex addition/removal, edge addition/removal

- Social Networks (hundreds of millions of users)
  - 10 vertices per second
  - 200 edges per second

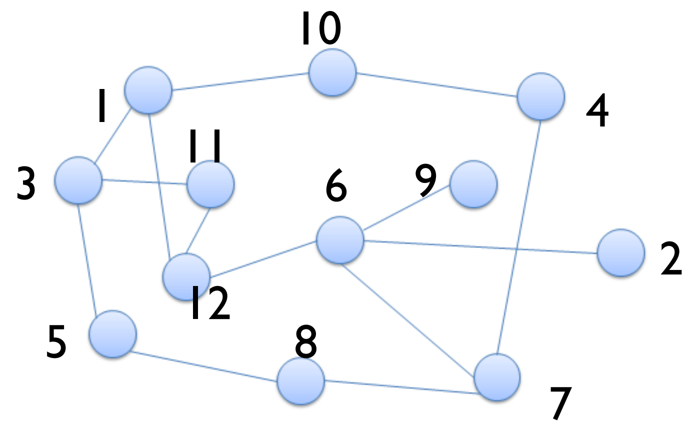
- Social GPS (millions of users)
  - 5 vertices per second
  - 2,000 edges per second
- The brain (hundred of billions of neurons)
  - 10,000 vertices per second
  - 200,000 edges per second

Network Representation



- A communication network is represented by a graph
- Vertices have unique IDs of size O(log n) each
- A messages traverses an edge within one round
- Running time = number of rounds to provide a solution
- Update time = number of rounds to update a solution

Network Models



Model #0 Static: Network does not change

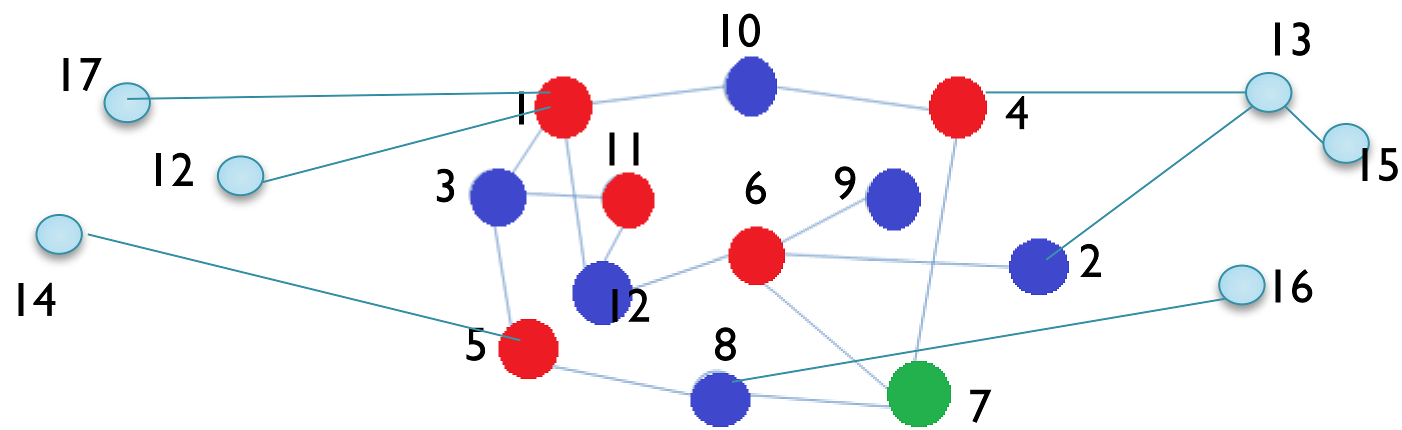
Model #1 Dynamic single change

Model #2 Dynamic restricted change

Model #3 Dynamic unrestricted change

Step-by-step changes

Network Models



Model #0 Static: Network does not change

Model #1 Dynamic single change

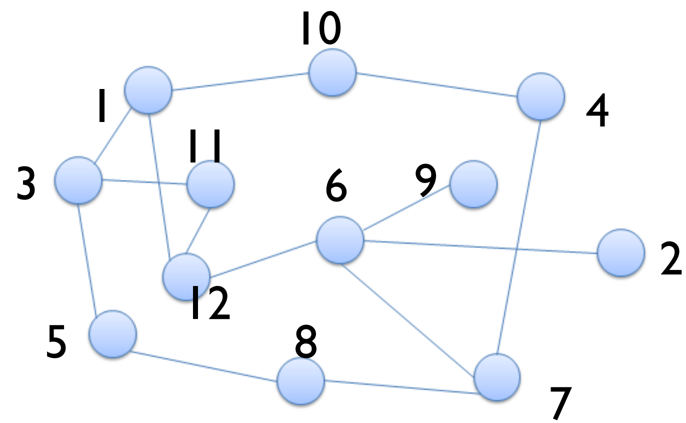
Model #2 Dynamic restricted change

Model #3 Dynamic unrestricted change

(Models #1–#3: Step-by-step changes)

Model #4 Dynamic changes during execution

Symmetry Breaking Problems

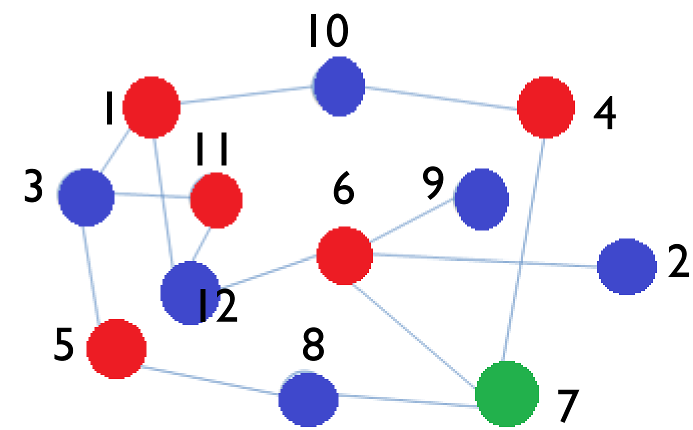


- Coloring

(Δ+1)–vertex-coloring , (2Δ–1)–edge-coloring, defective-coloring,…

- Maximal Independent Set (MIS)
- Maximal Matching (MM)

Symmetry Breaking Problems



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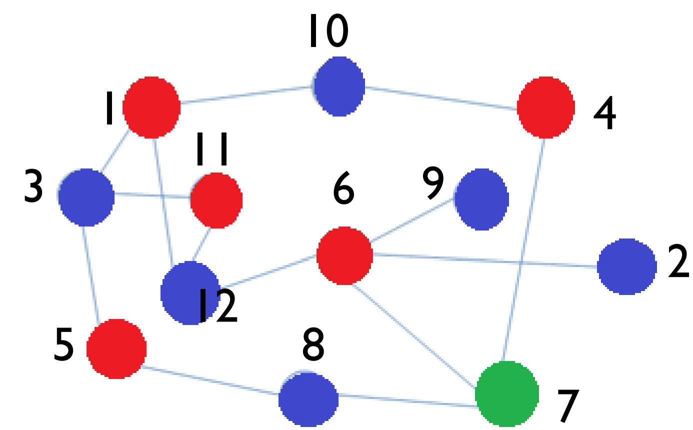

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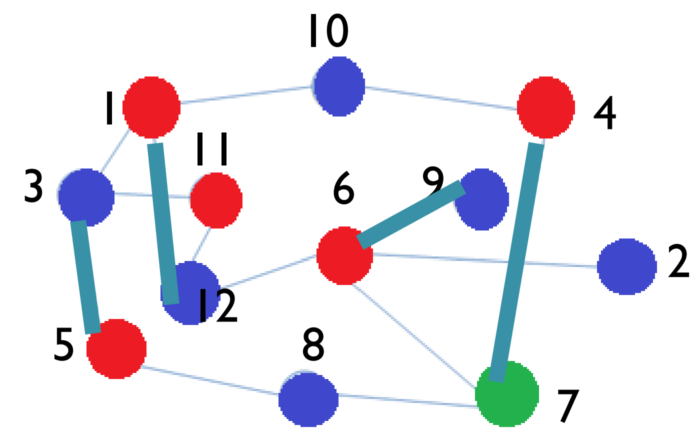

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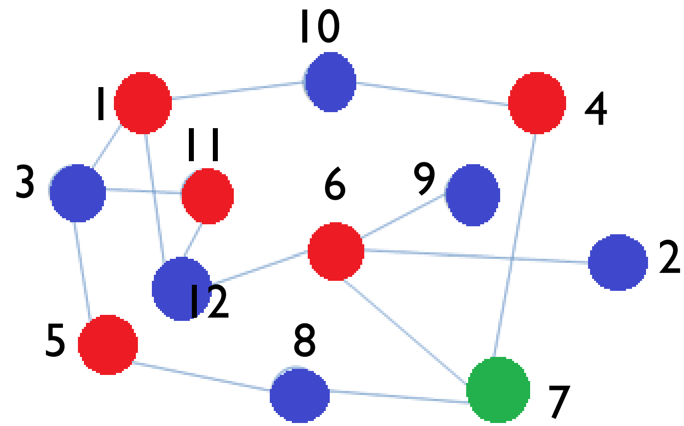
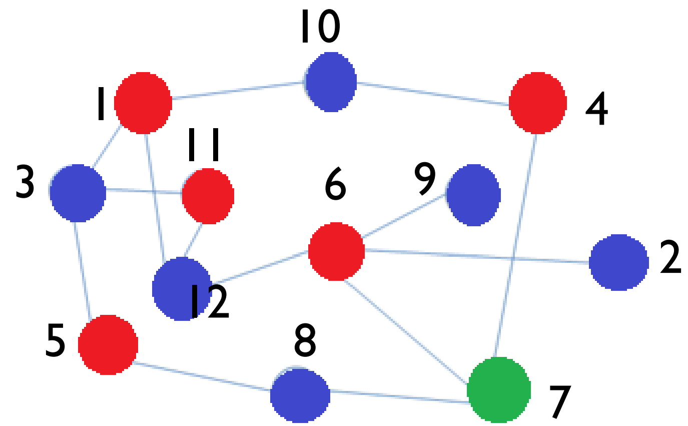


Coloring, MIS and MM belong to the class of

locally-checkable problems

(Local Decision Class, Fraigniaud, Korman and Peleg 2011)

Dynamic Single Change - Coloring



Local fixing in O(1) rounds

König and Wattenhofer 2013

- Adding a vertex or an edge
- Removing a vertex or an edge

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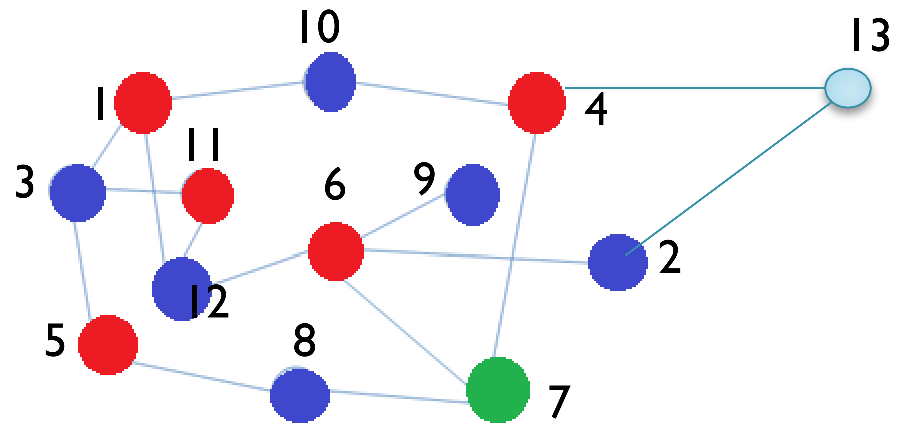

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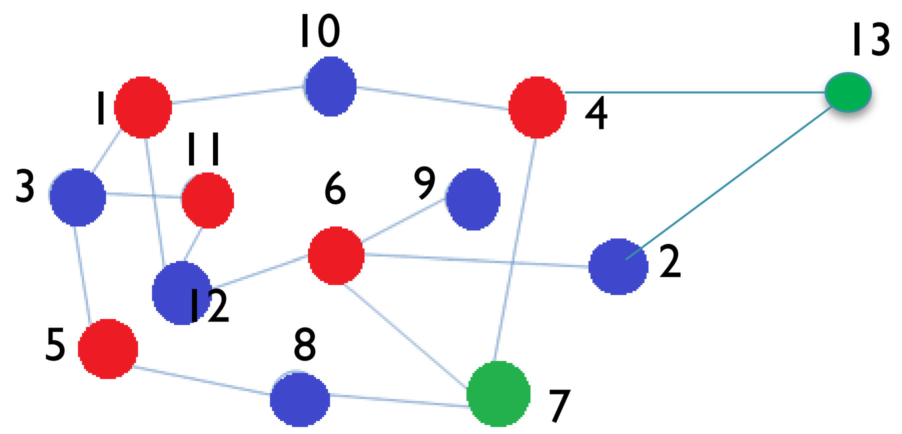

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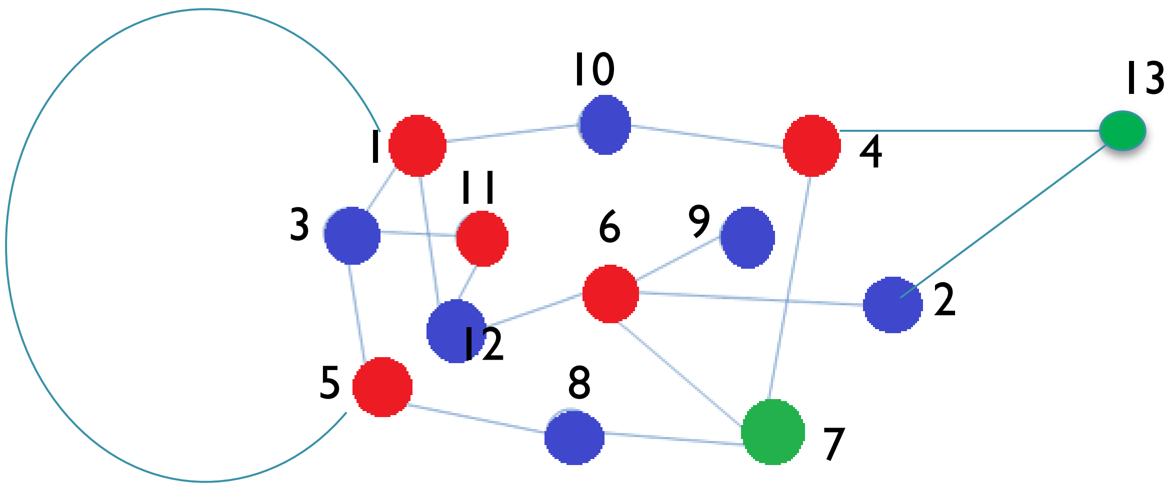

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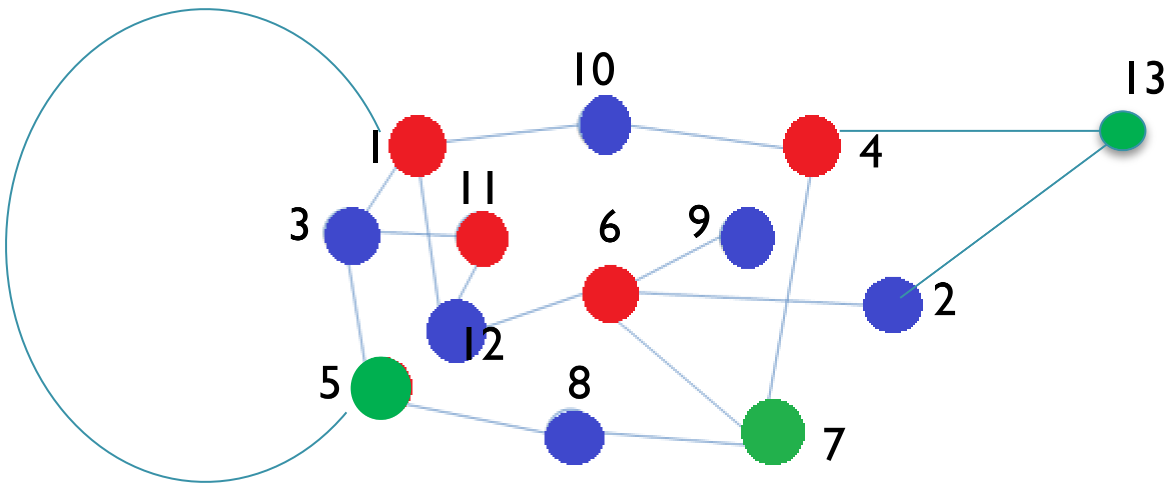

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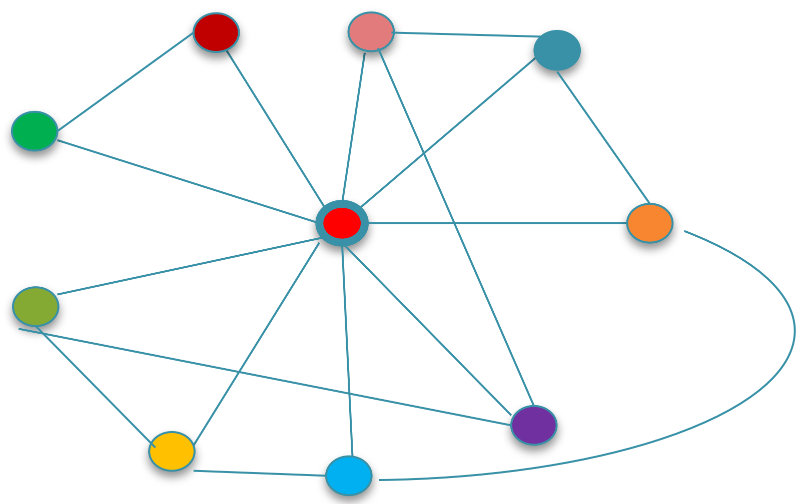

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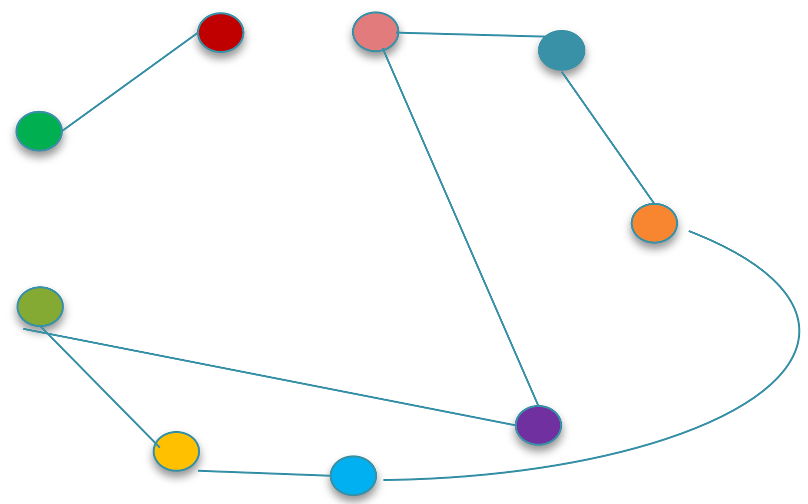

Local fixing in O(1) rounds

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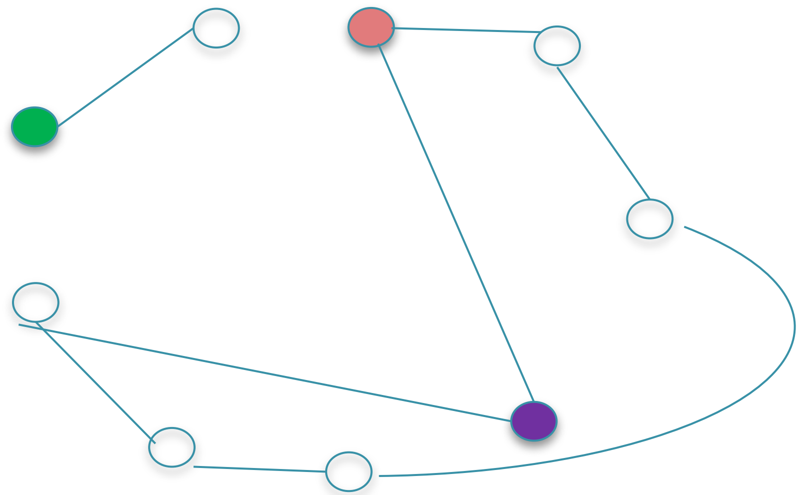
This is a proper coloring, but is it a **(Δ+1)-coloring?**

Dynamic Single Change - Coloring

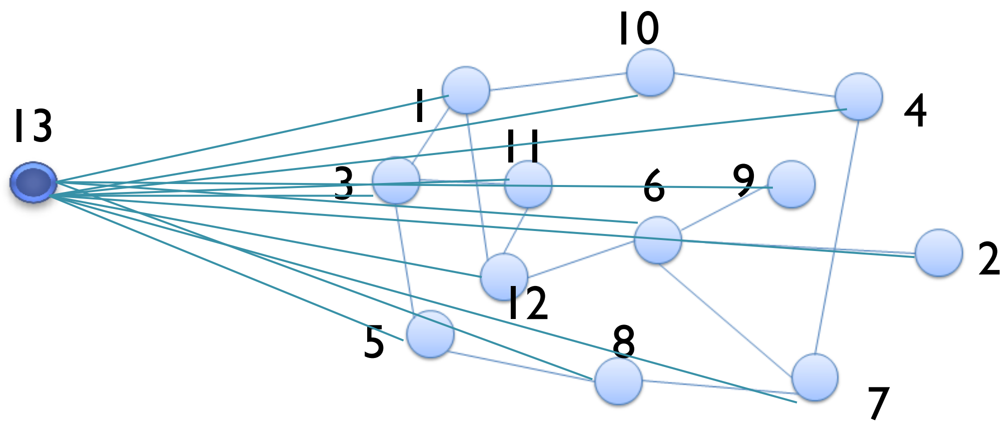


Possible solution:

Delete all colors out of range $\{1,2,\ldots,\Delta+1\}$, recompute solution for colorless vertices.

If a vertex leaves "gracefully" then O(1)-time solution is possible

Dynamic Single Change - MIS

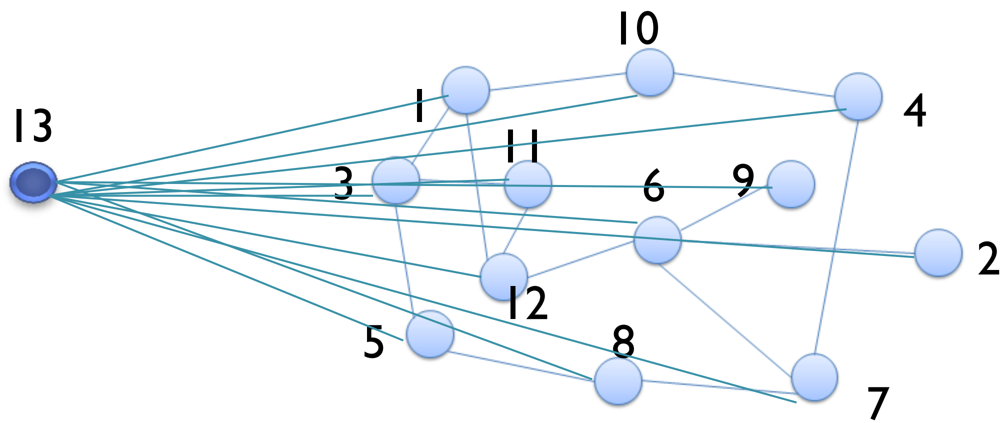


An MIS may consist of a single vertex.

Vertex removal may require recomputation for the entire graph.

If a vertex leaves "gracefully", it can communicate new solution within O(1) rounds.

Dynamic Single Change - MIS

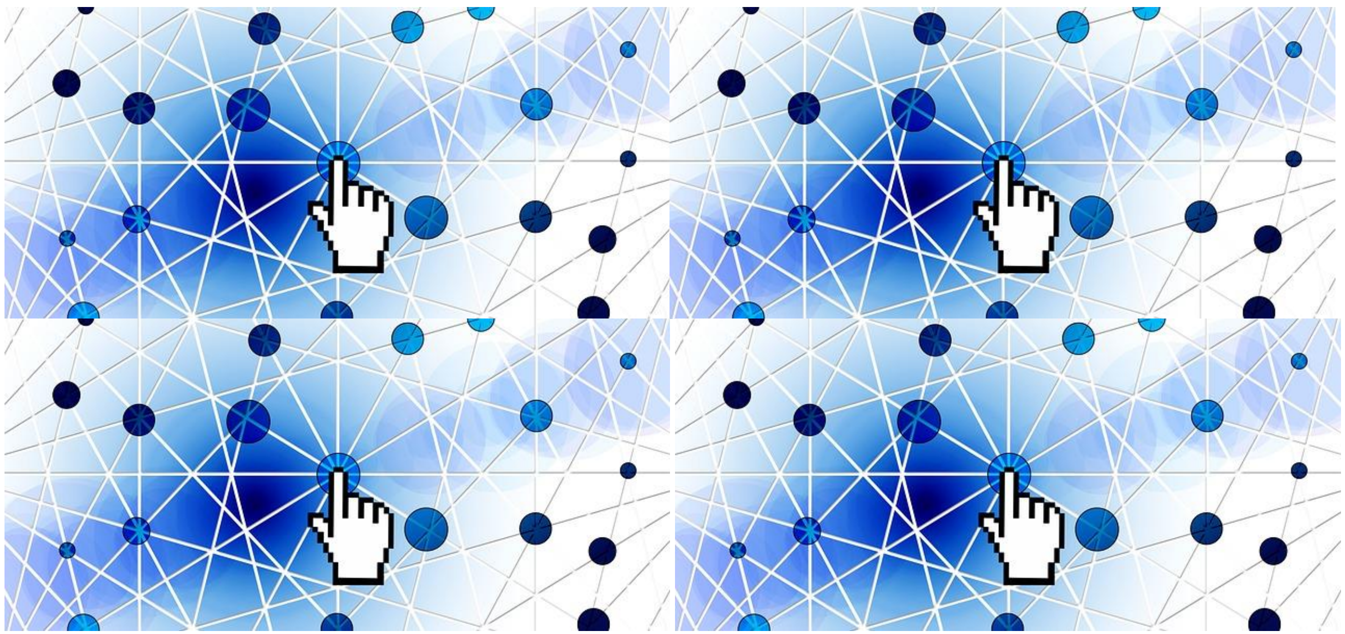


What if vertices do not leave “gracefully”?

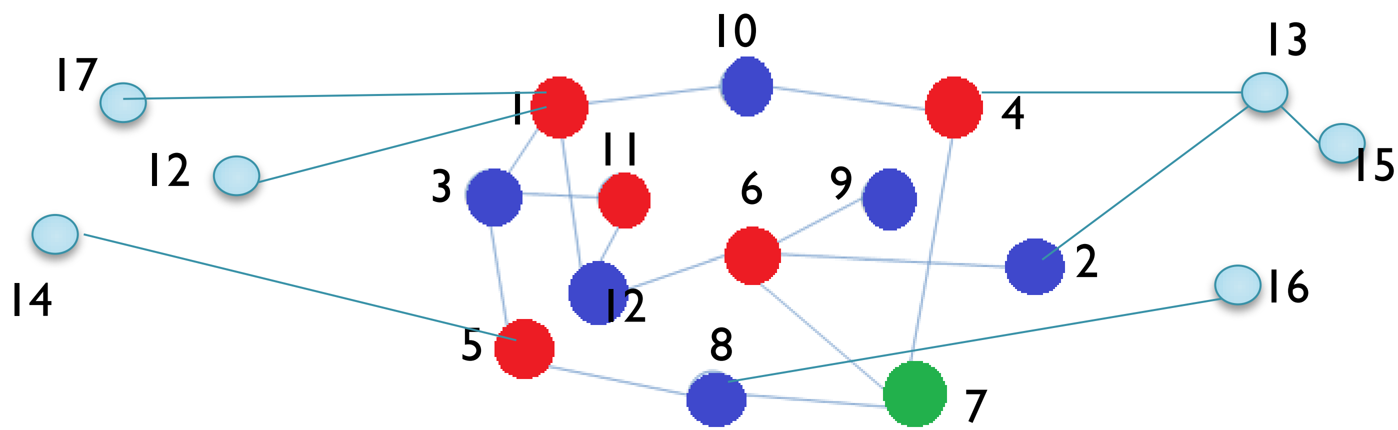
- Expected O(1)-time solution
  Censor-hillel, Haramaty and Karnin 2016

  Simulation of a greedy sequential MIS
  with a random ordering.

Dynamic Unrestricted Change



Static Graphs with Partial Solution



<u>*Theorem:*</u>

Suppose that we have a **static algorithm** for a **locally-checkable** problem on graphs with **partial solution** with **time T**.

Then we have a **dynamic algorithm** for the problem with **update time T**.

Obtaining Dynamic Algorithms

Static Algorithm

↓

Static Algorithm
for
Partial Solution

↓

Dynamic Algorithm

Obtaining Dynamic Algorithms

Static Algorithm

↓

Static Algorithm
for
Partial Solution

↓

Dynamic Algorithm

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Dynamic Algorithm

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Static Algorithm
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Partial Solution

↓

Static Algorithm

Static $O(\Delta^2)$-Coloring

Linial 1987

Running time: O(log* n).

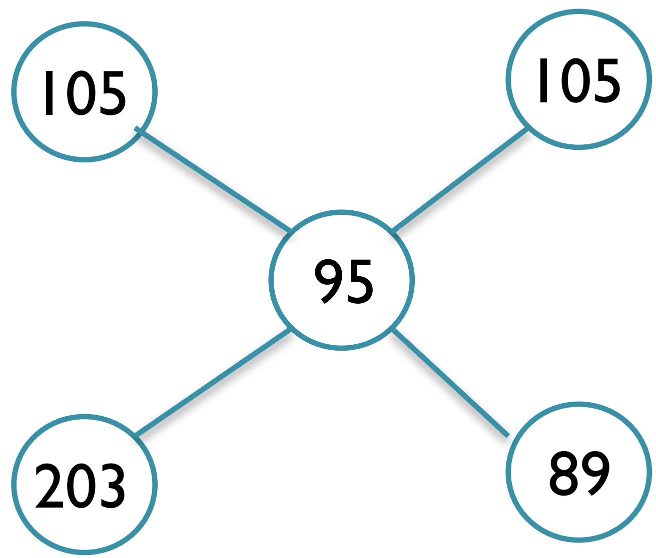
Very high-level description:

1. Initial n-coloring is obtained using IDs

2. In each round the number of colors is reduced from k to $O(\Delta^2 \log k)$.

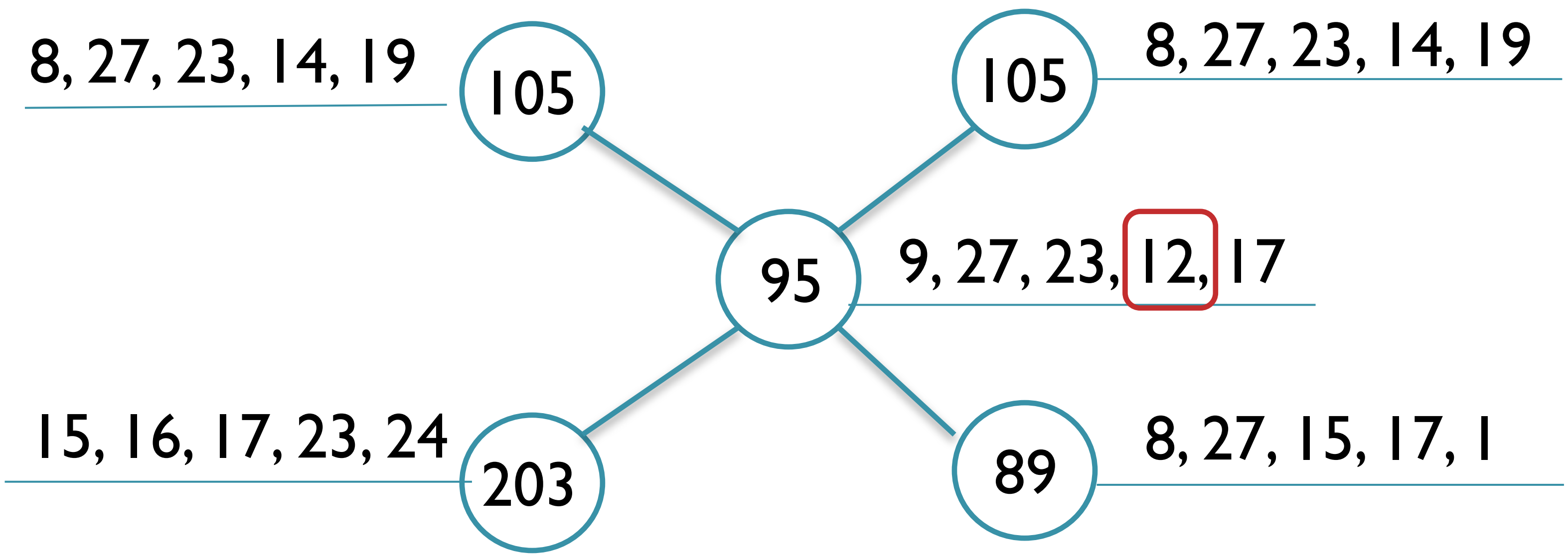
$$n \rightarrow \Delta^2 \log n \rightarrow \Delta^2(\log \Delta + \log\log n) \rightarrow \cdots \rightarrow \Delta^2 \log \Delta \rightarrow \Delta^2$$

Static $O(\Delta^2)$-Coloring



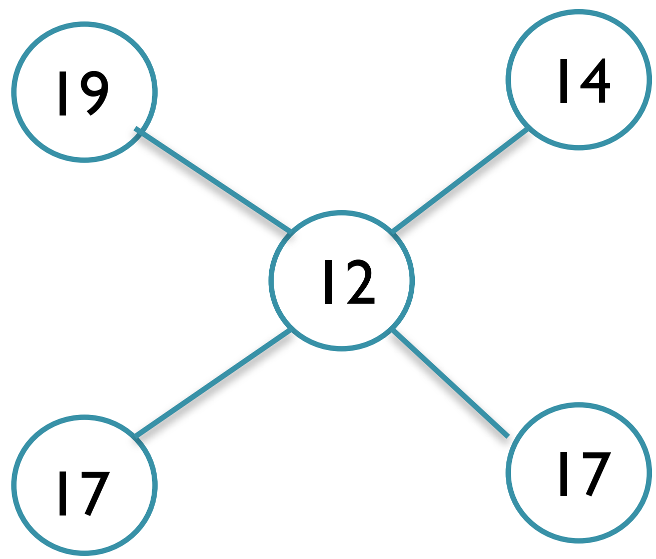
- Each vertex constructs a list of colors using its current color

Static $O(\Delta^2)$-Coloring



- Each vertex constructs a list of colors using its current color
- Each list must have a color that does not appear in the neighbors lists

Static $O(\Delta^2)$-Coloring



- Each vertex constructs a list of colors using its current color
- Each list must have a color that does not appear in the neighbors lists

This color is selected as the new color. New coloring is proper!

Implementing One Round

$O(\Delta^3)$ colors $\rightarrow$ $O(\Delta^2)$ colors

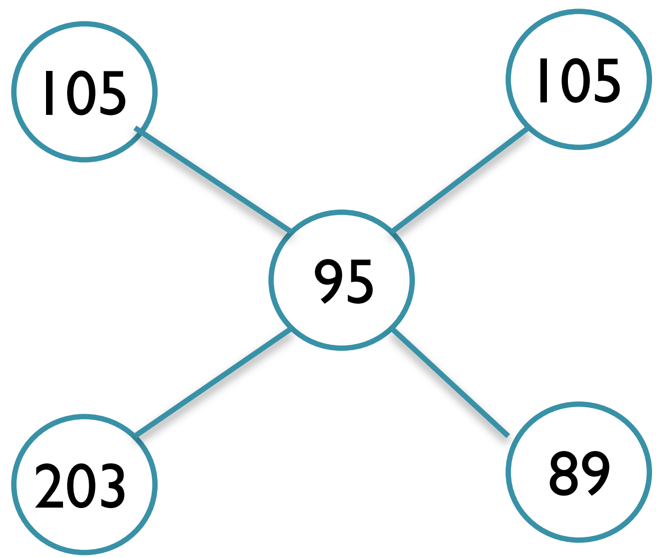
Let $q = O(\Delta)$ be a prime,
such that the number of colors is at most q^3.

There are q^3 distinct polynomials over the field Z_q:

$$a + bx + cx^2 \qquad\qquad 0 \leq a, b, c \leq q - 1$$

Each of the q^3 colors is assigned a distinct polynomial.

Implementing One Round



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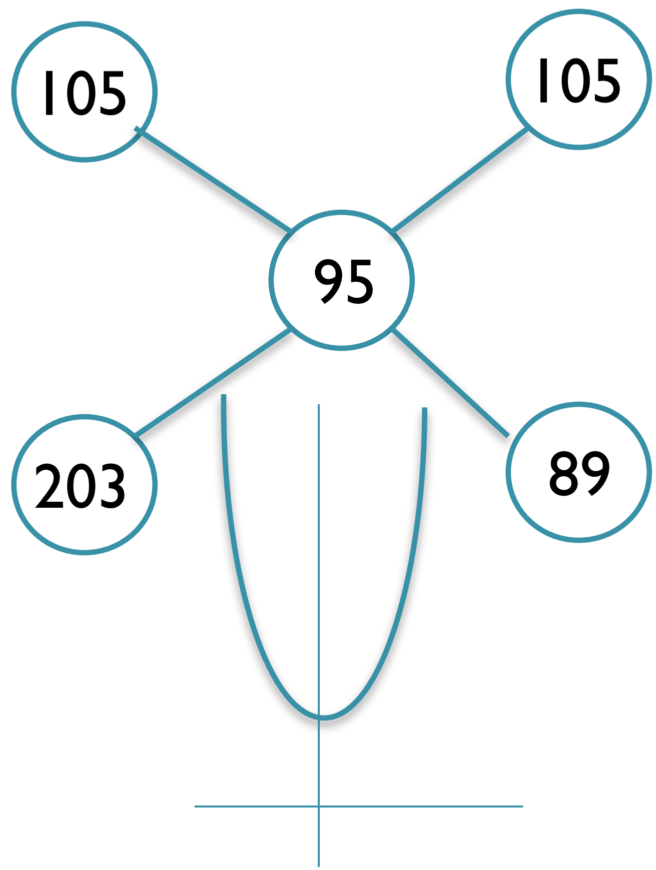

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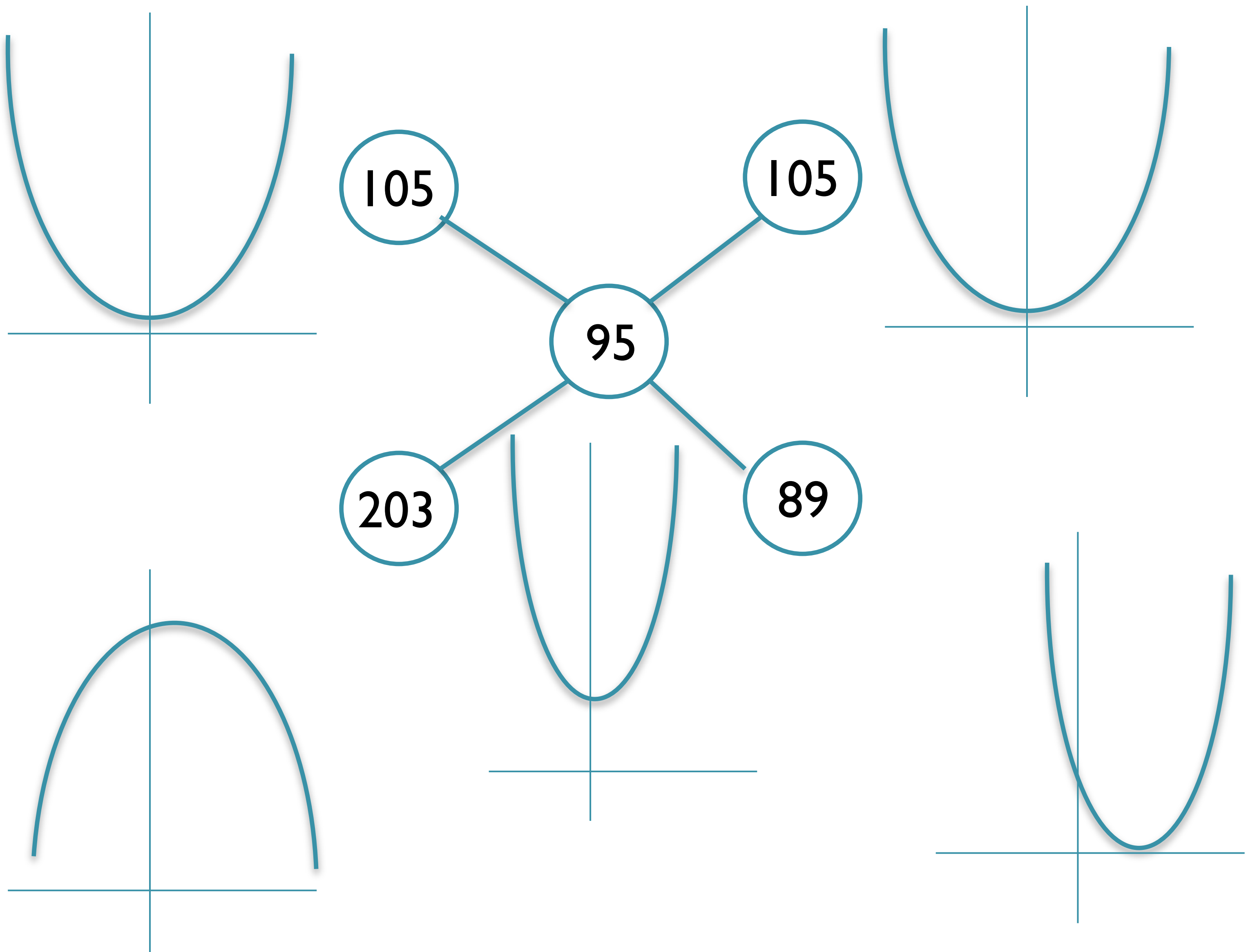

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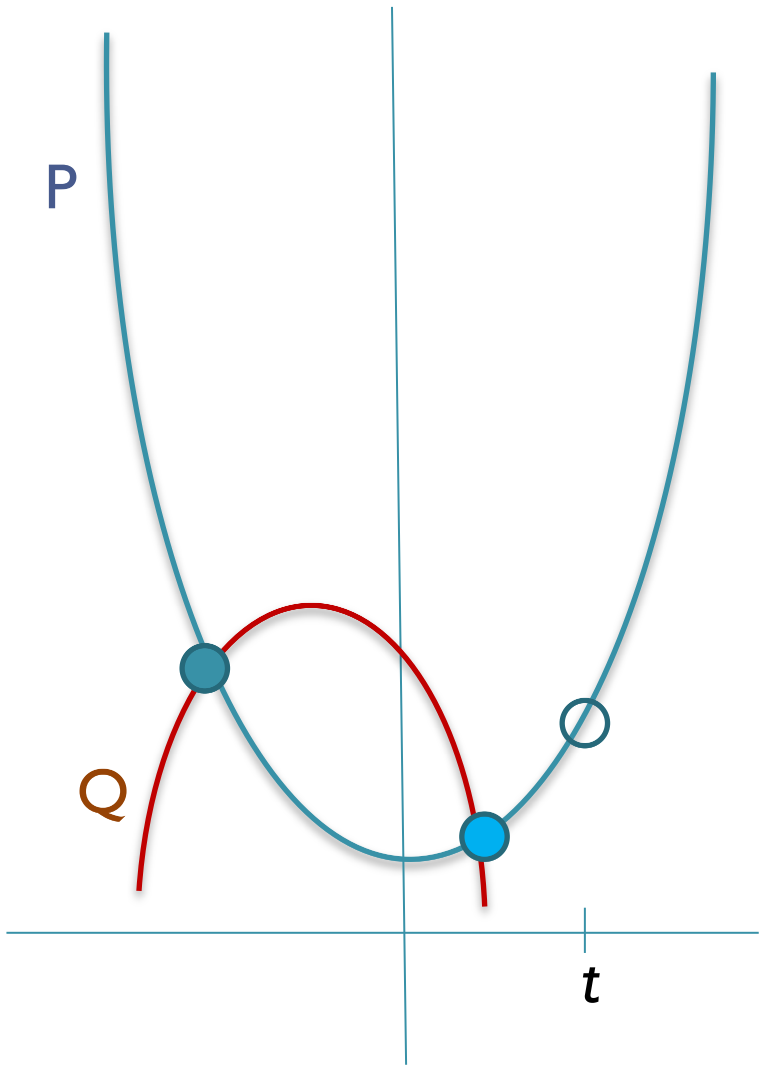


For each vertex:

- At most 2 intersections with each neighbor
- At most 2Δ intersections with all neighbors

Choose $q \geq 2\Delta + 1$

There is $t, 0 \leq t \leq q - 1$:

$$< t, P(t) > \neq < t, Q(t) >$$

for all neighbors' Q.

Implementing One Round

There is $t, 0 \leq t \leq q-1$:
$< t, P(t) > \neq < t, Q(t) >$

for all neighbors' Q.

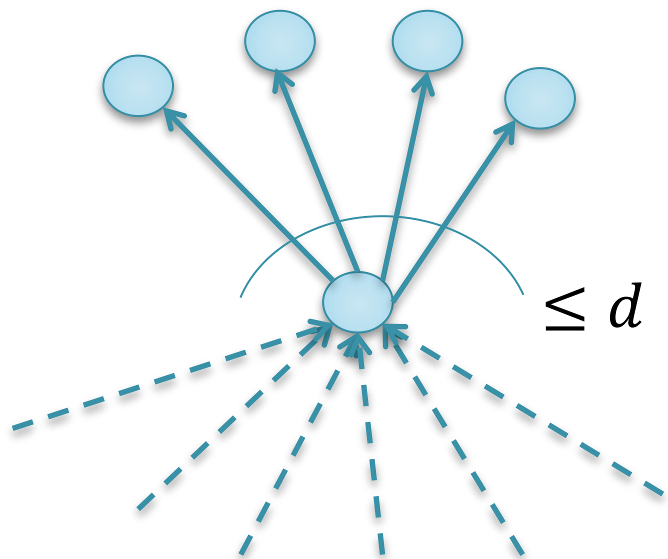
$< t, P(t) >$ is the new color.

For each pair of neighbors: $< t, P(t) > \neq < r, Q(r) >$

Number of colors: $q^2 = O(\Delta^2)$.

Using less than Δ^2 Colors

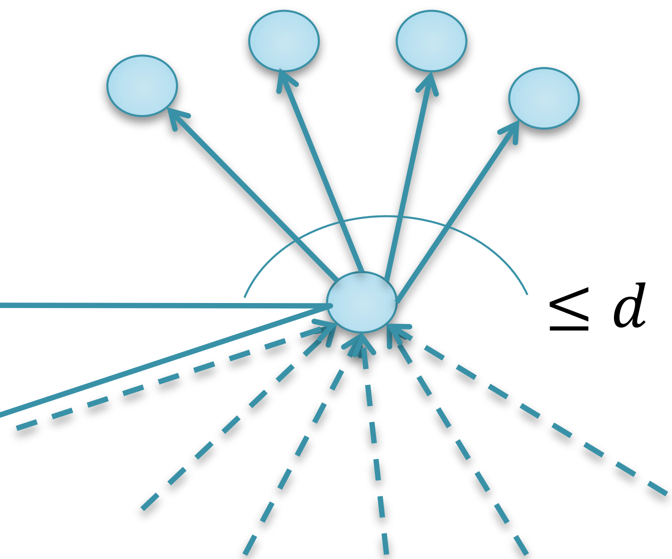
Suppose we have an orientation with out-degree d



Using less than Δ^2 Colors

Suppose we have an orientation with out-degree d

Look only on outgoing neighbors. Select a color that is not in their lists.



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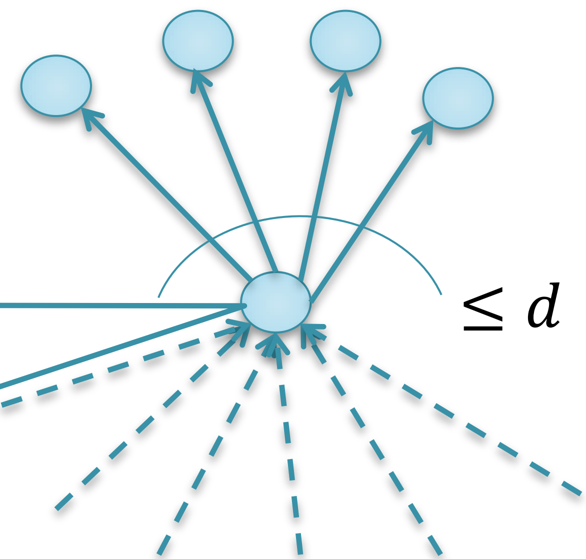

$O(d^2)$-coloring is computed in $O(\log^* n)$ time.

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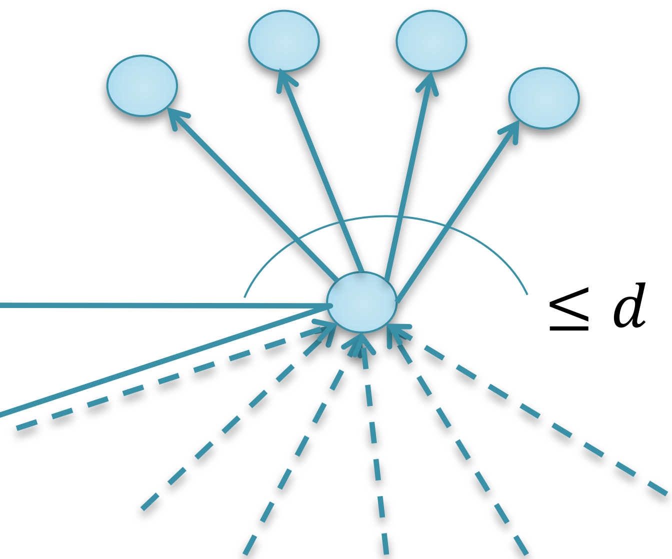

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Arboricity a is the minimum number of forests.

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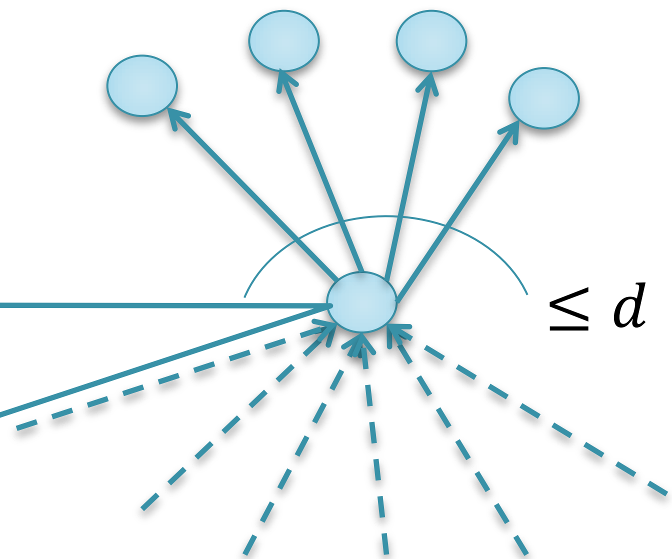

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Arboricity a is the minimum number of forests.

$O(a)$-orientation in $O(\log n)$ time. Barenboim and Elkin 08.

Orientations with Small Out-Degree

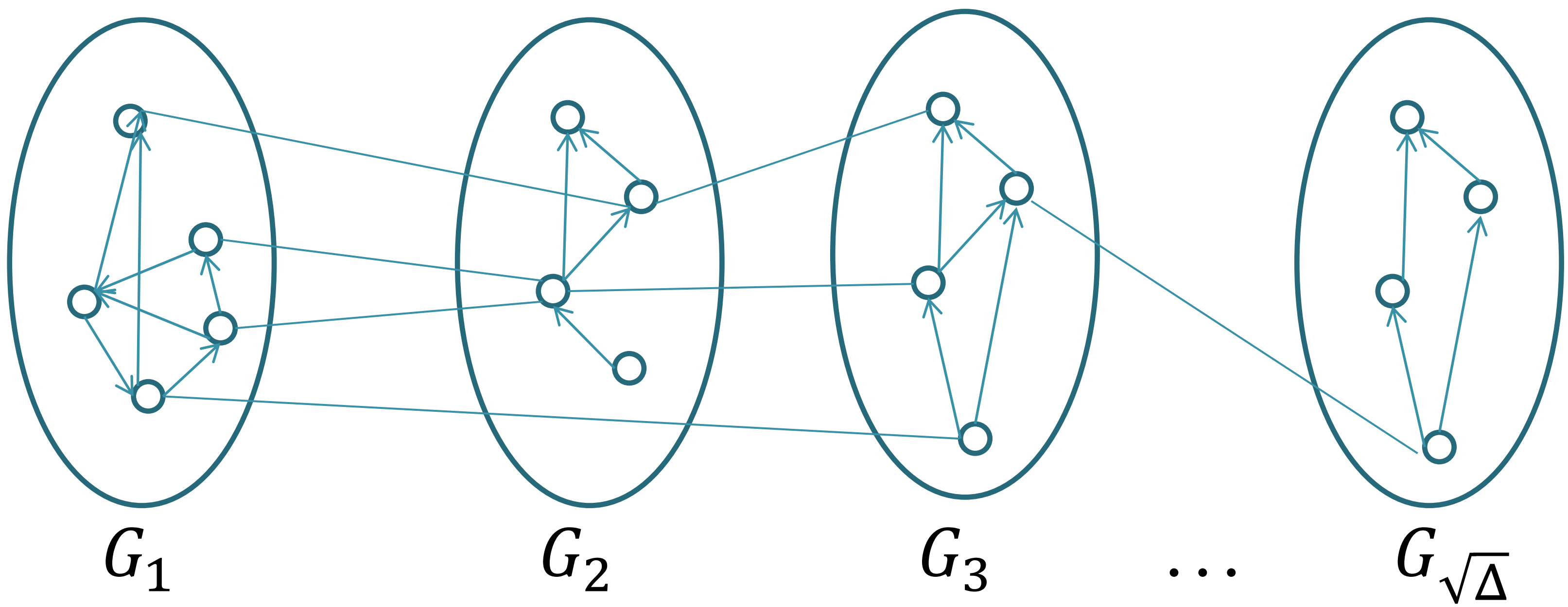
If we have an orientation with $d \leq \sqrt{\Delta}$,
we can compute $O(\Delta)$-coloring in $O(\log^* n)$ time!

Small out-degree orientation does not always exist. ☹

Partition the graph into $\sim\sqrt{\Delta}$ vertex-disjoint subgraphs,
each subgraph with out-degree $O\left(\sqrt{\Delta}\right)$.

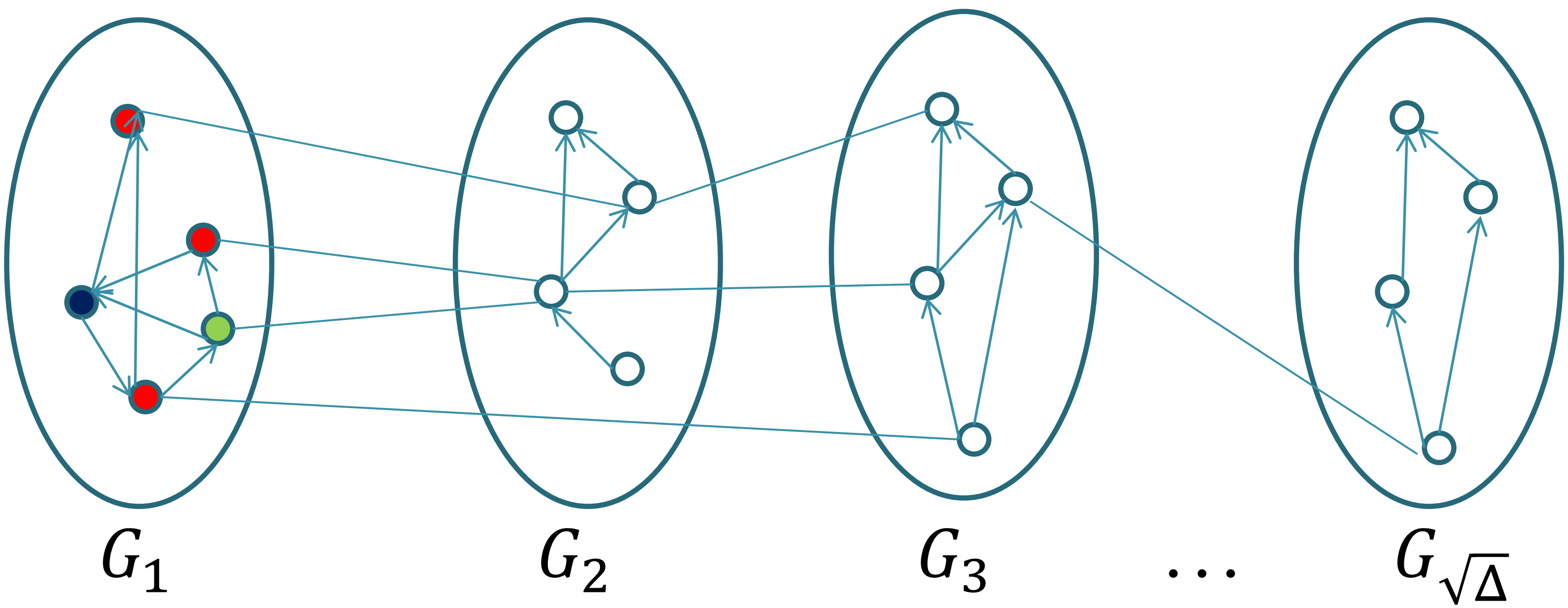
Color subgraphs one by one - $O(\log^* n)$ time per subgraph. ☺

Graph Partition



Graph Partition

Each subgraph is properly colored.



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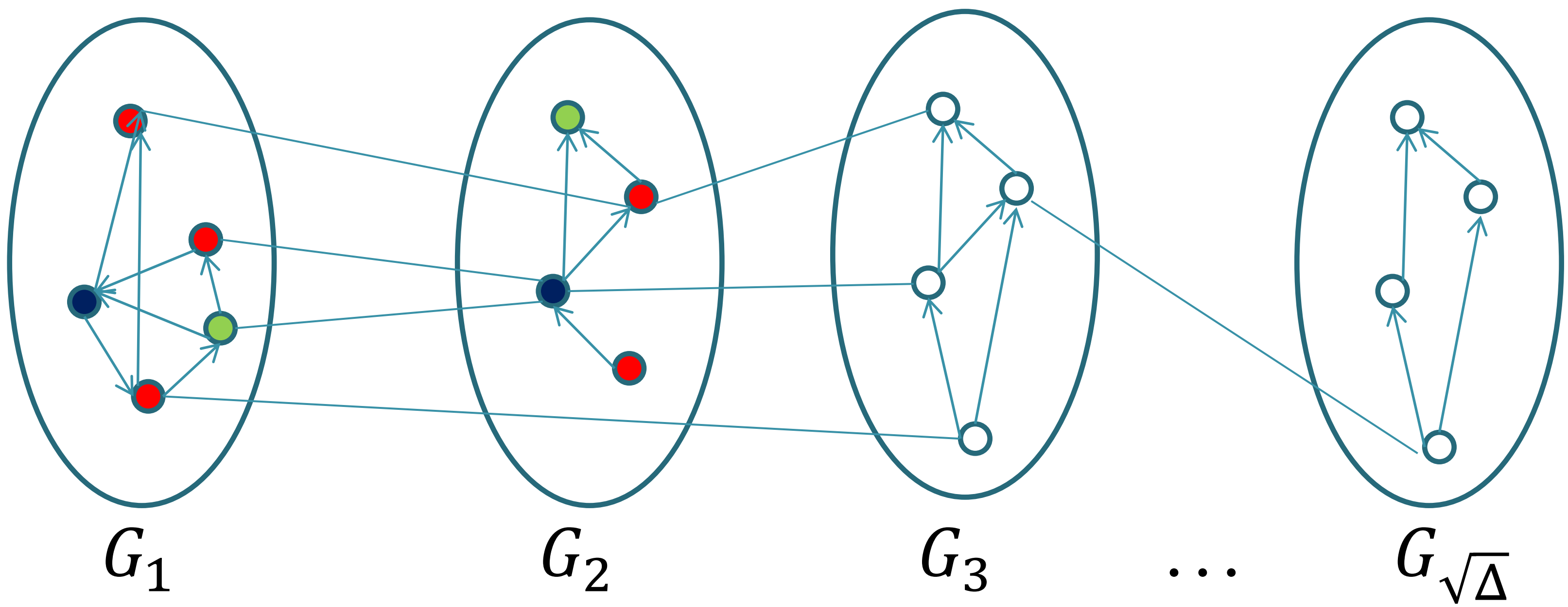

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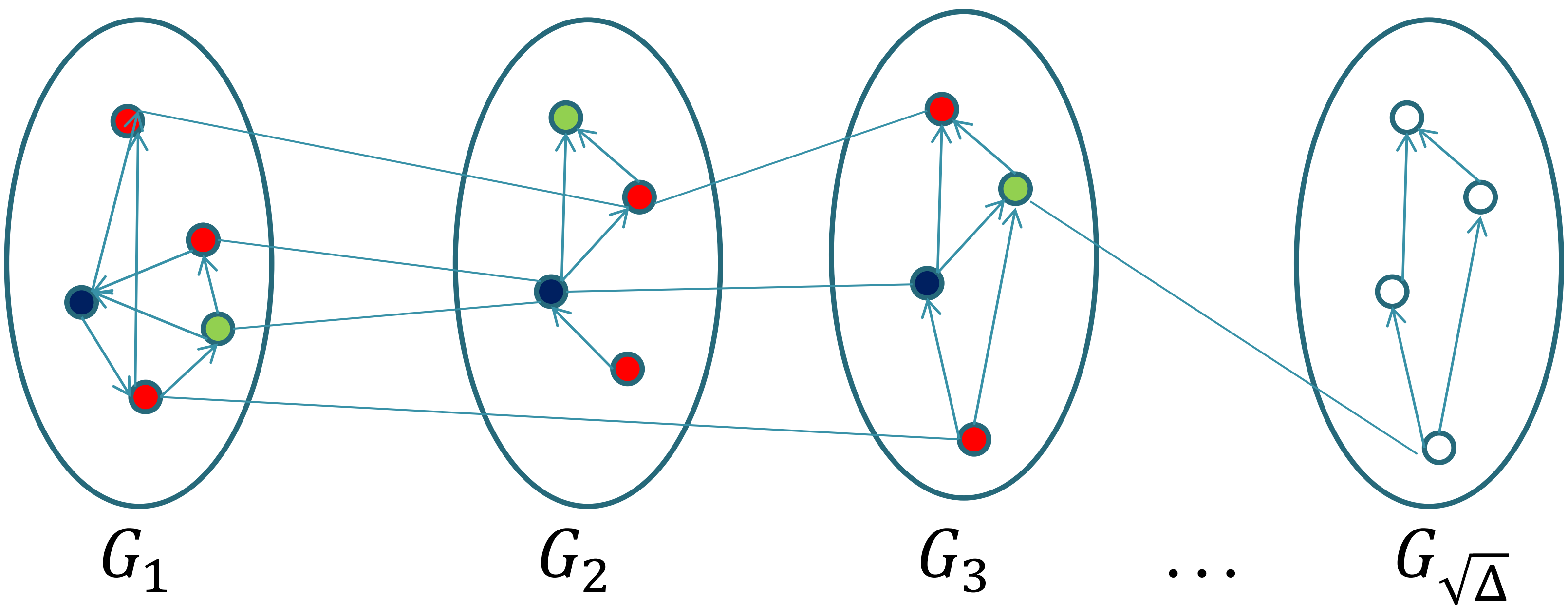

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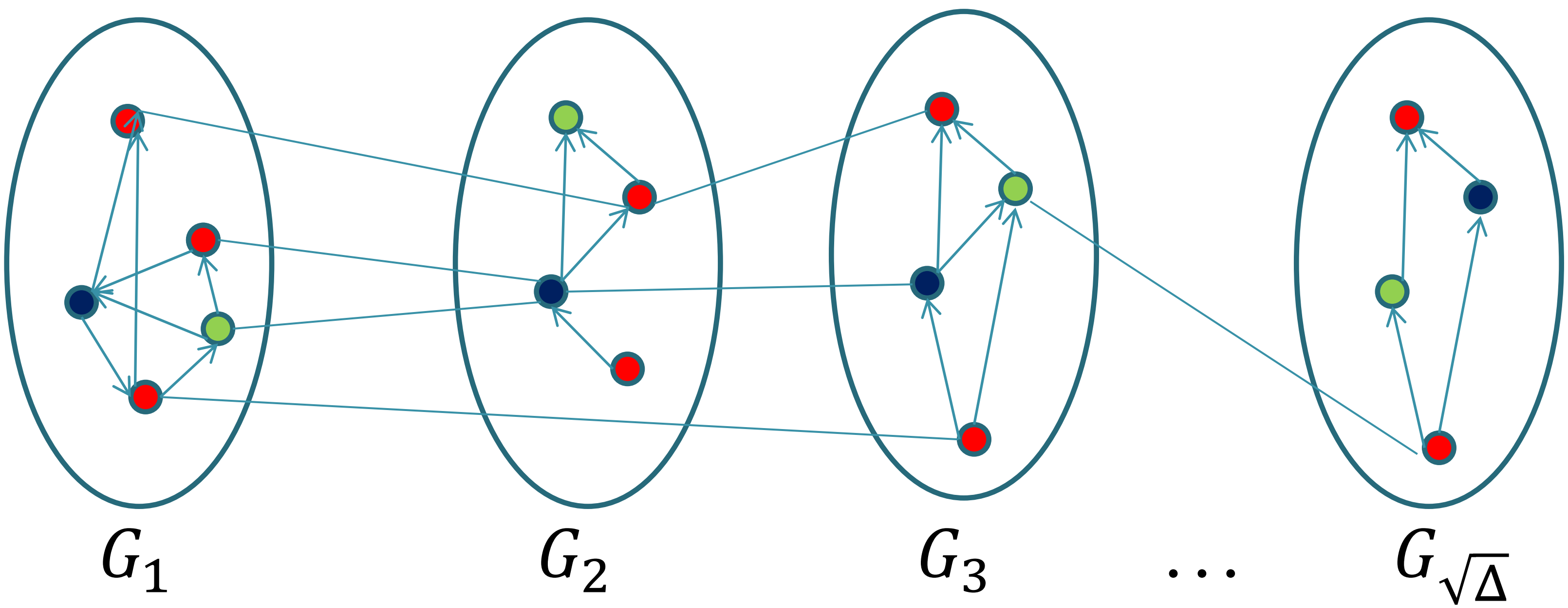
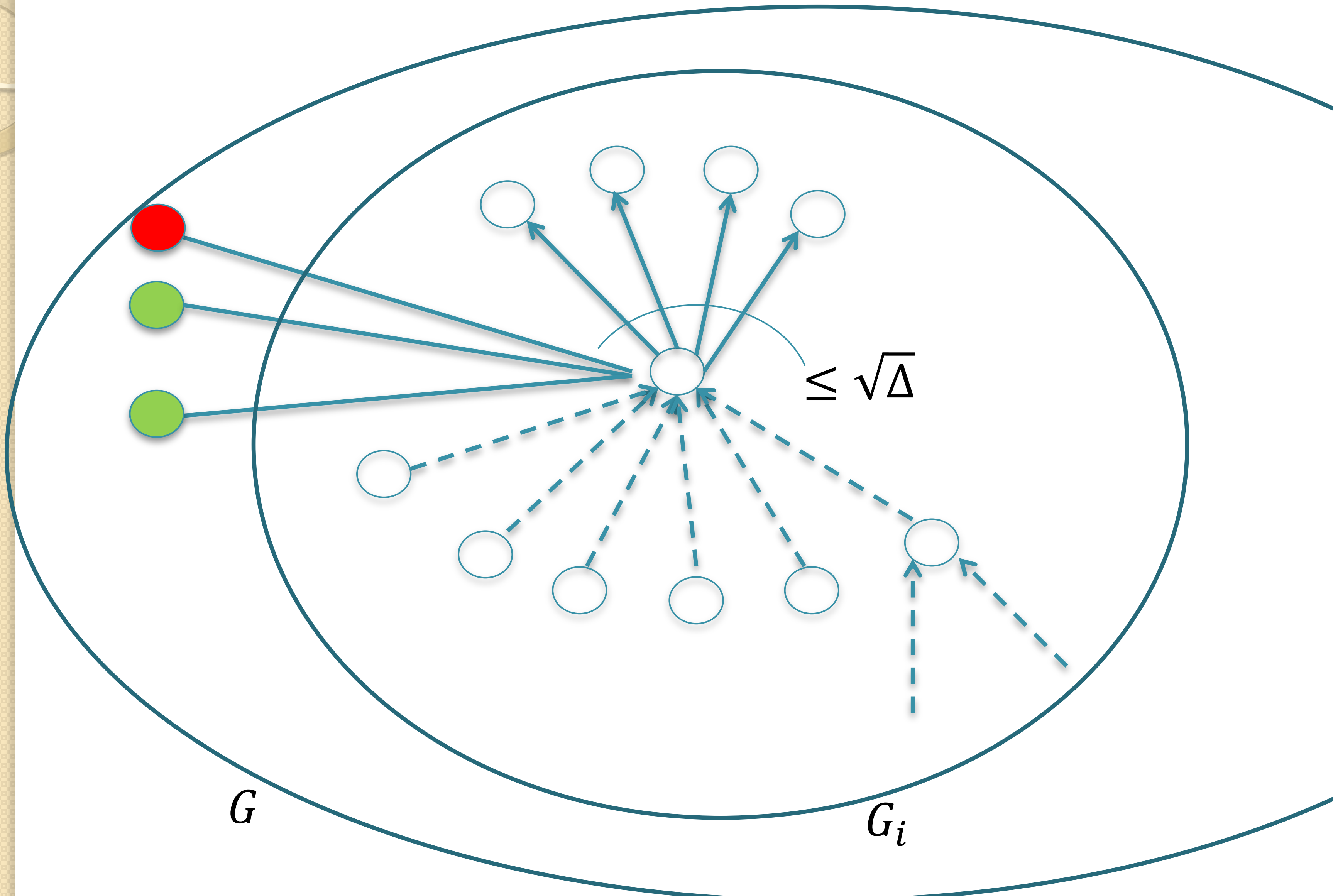


Problem: monochromatic edges between subgraphs.

Solution: make it work in **partially colored** graphs.

Coloring Partially-Colored Graphs



Coloring Partially-Colored Graphs

Barenboim 2015

Each vertex may have up to Δ colored neighbors.

Each color is a forbidden coordinate $< x, f(x) >$.

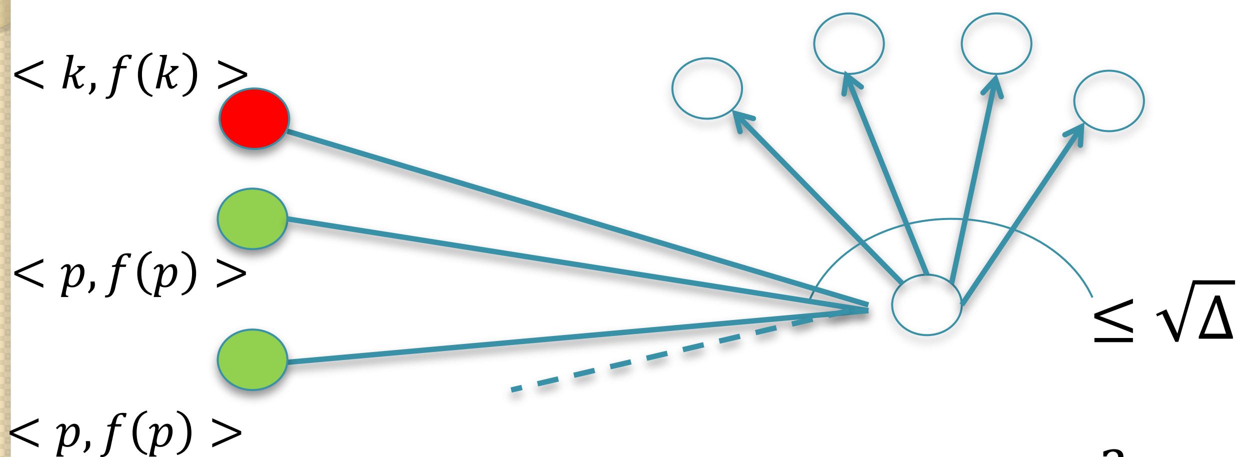
Problem: The size of the field is only $O(\sqrt{\Delta})$.

Solution:

Each vertex defines $O(\sqrt{\Delta})$ non-intersecting polynomials.

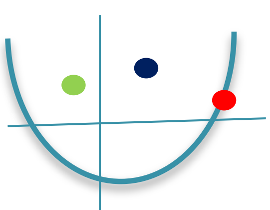
Then we can find a polynomial with a good coordinate.

Coloring Partially-Colored Graphs



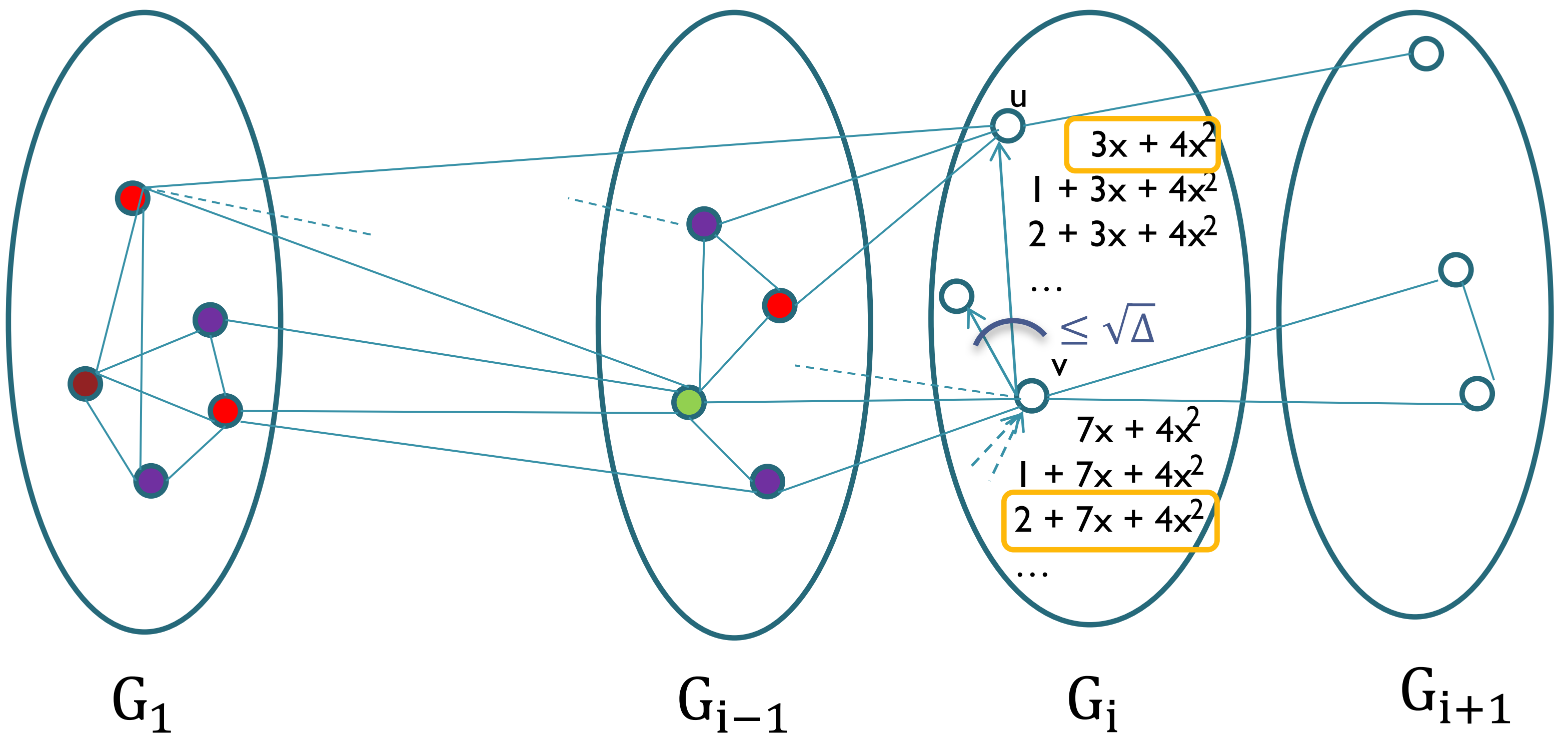
$$ax + bx^2$$
$$1 + ax + bx^2$$
$$\boxed{2 + ax + bx^2}$$
$$\ldots$$
$$q - 1 + ax + bx^2$$

Find a polynomial with minimum number of conflicts



$$\sqrt{\Delta} \leq q = O(\sqrt{\Delta})$$

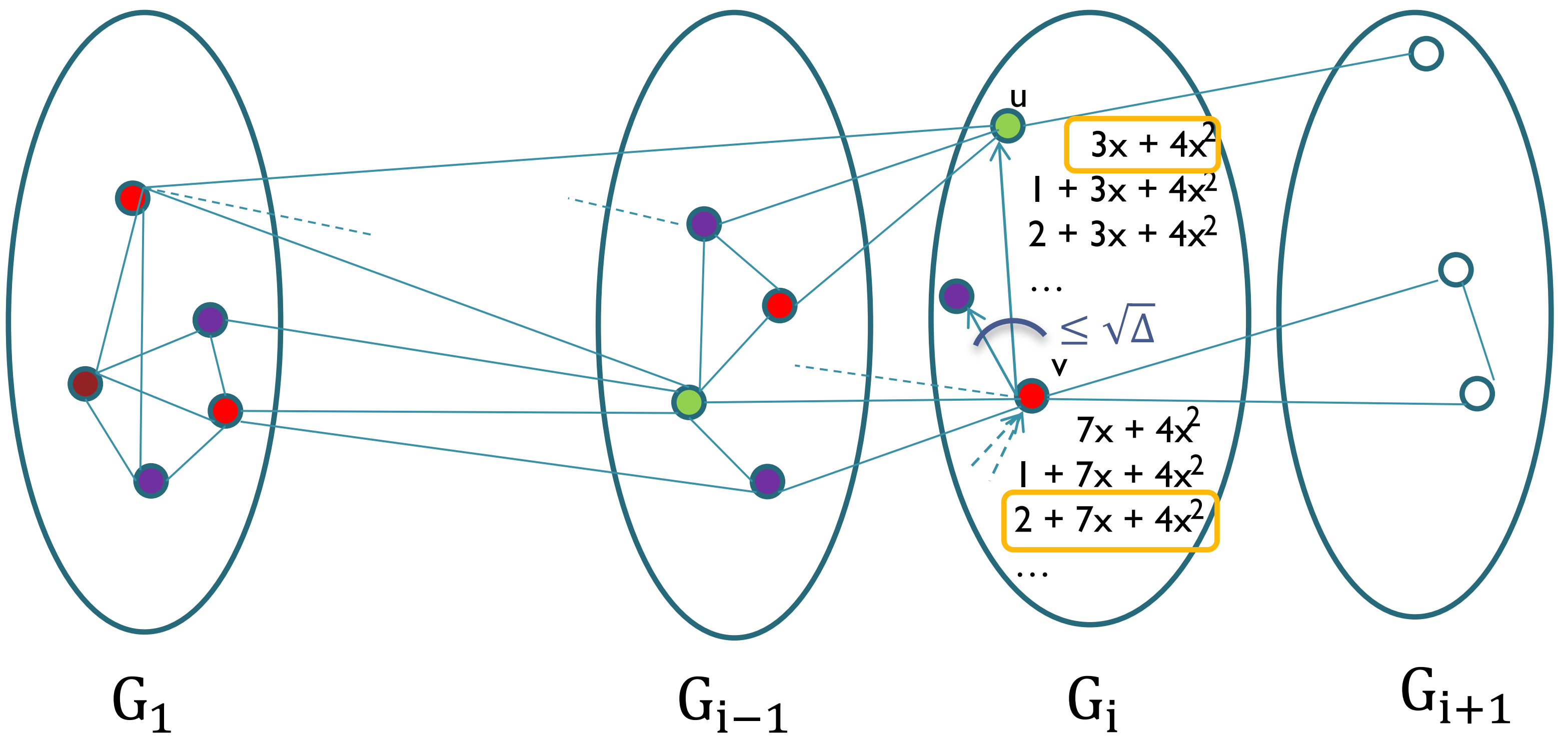
Coloring Partially-Colored Graphs



How to determine the coefficients a and b?

Using a helper temporary $O(\Delta)$-coloring of G_i.

Coloring Partially-Colored Graphs



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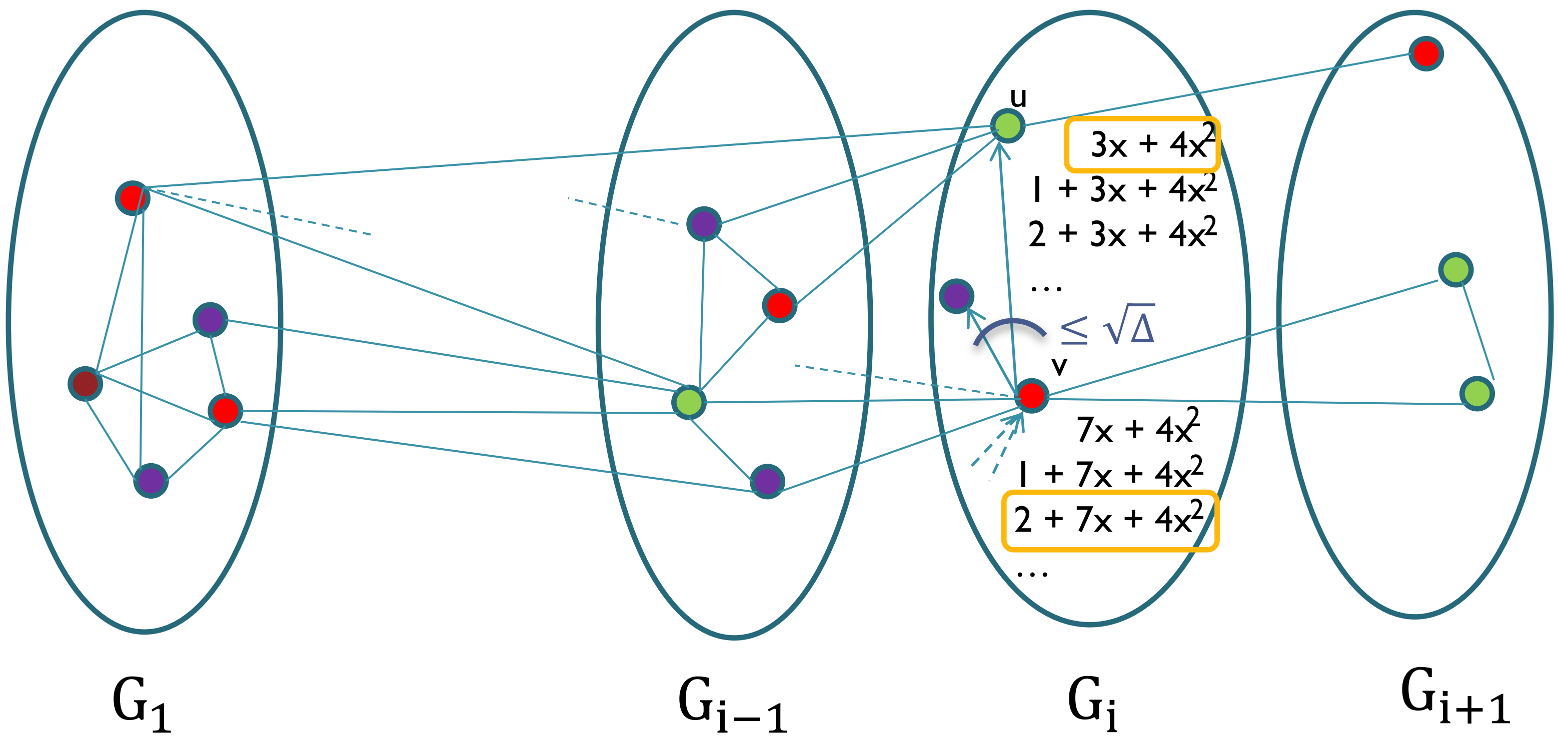

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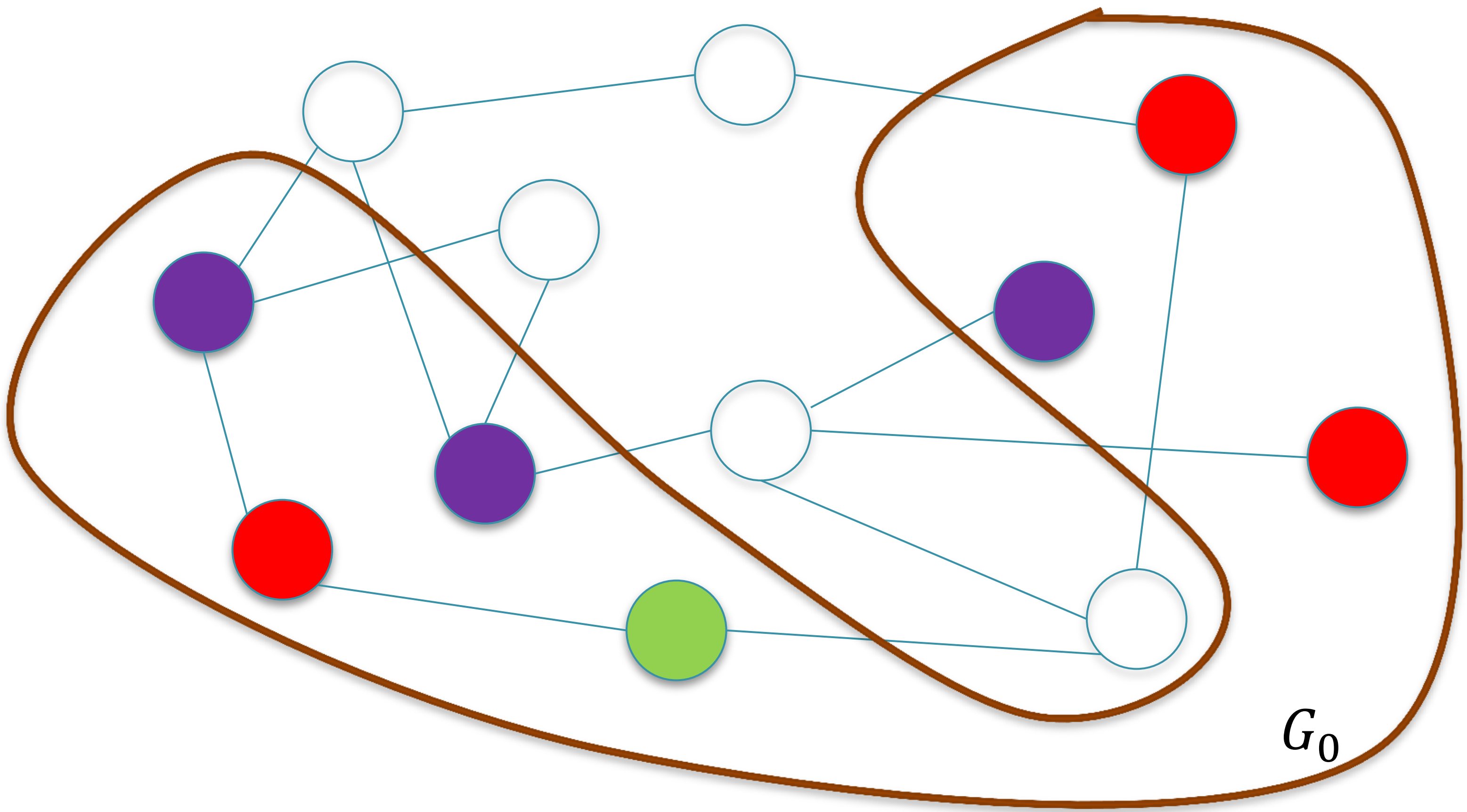

- Let $G_0 = (V_0, E_0)$ denote the subgraph of colored vertices
- Execute our algorithm on $V \setminus V_0$, and avoid conflicts with V_0.

Dynamic Algorithm

In each step (addition of vertices or edges, removal of vertices or edges) :

1. Perform local fixing to obtain a partial solution

2. Invoke static algorithm for partial solution

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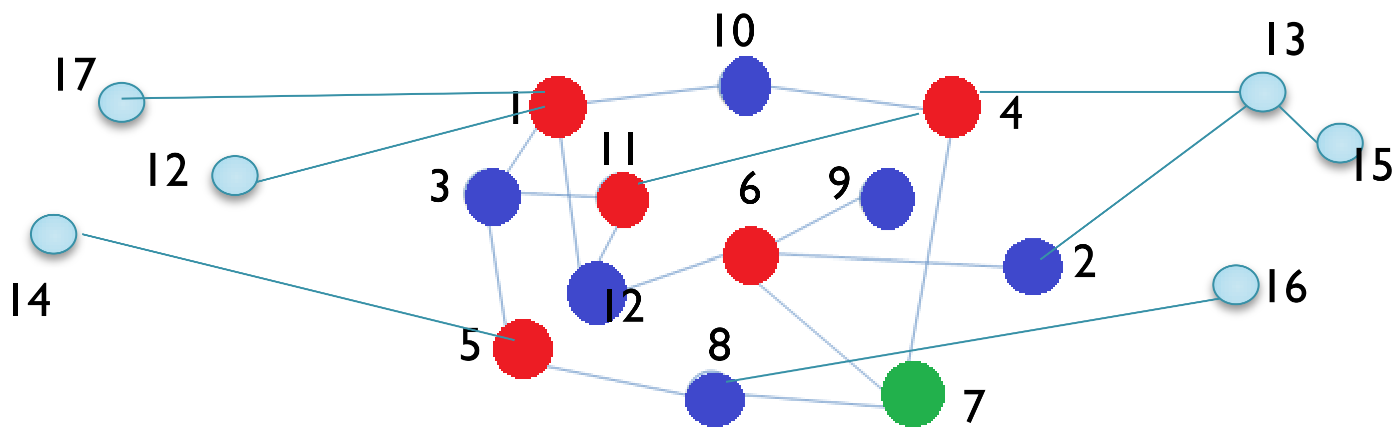

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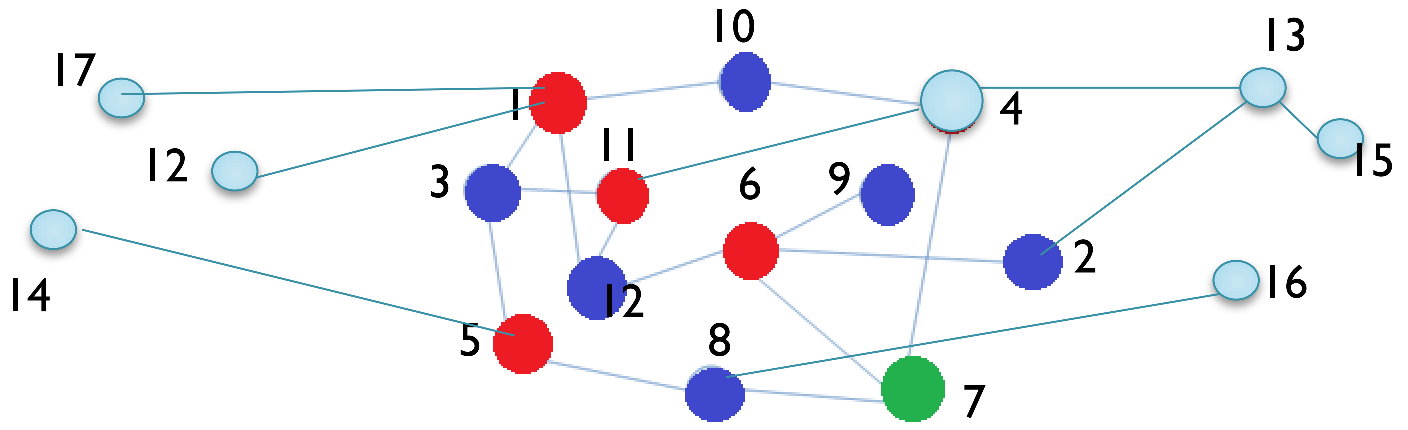

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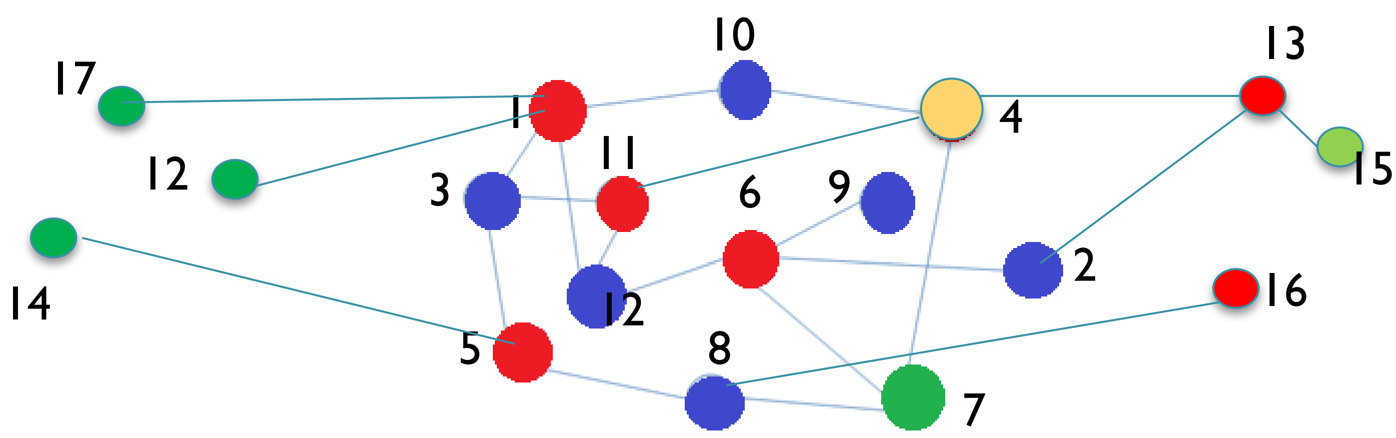
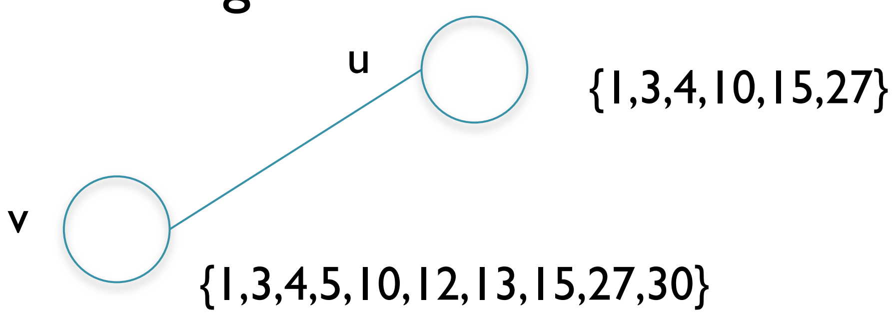


Static Algorithm for List-Coloring

<u>Input:</u>
Each vertex receives as input a list of at least $\Delta+1$ colors from a range of size $D = O(\Delta)$.

<u>Output:</u>
Each vertex selects a color from its list to obtain a proper coloring.

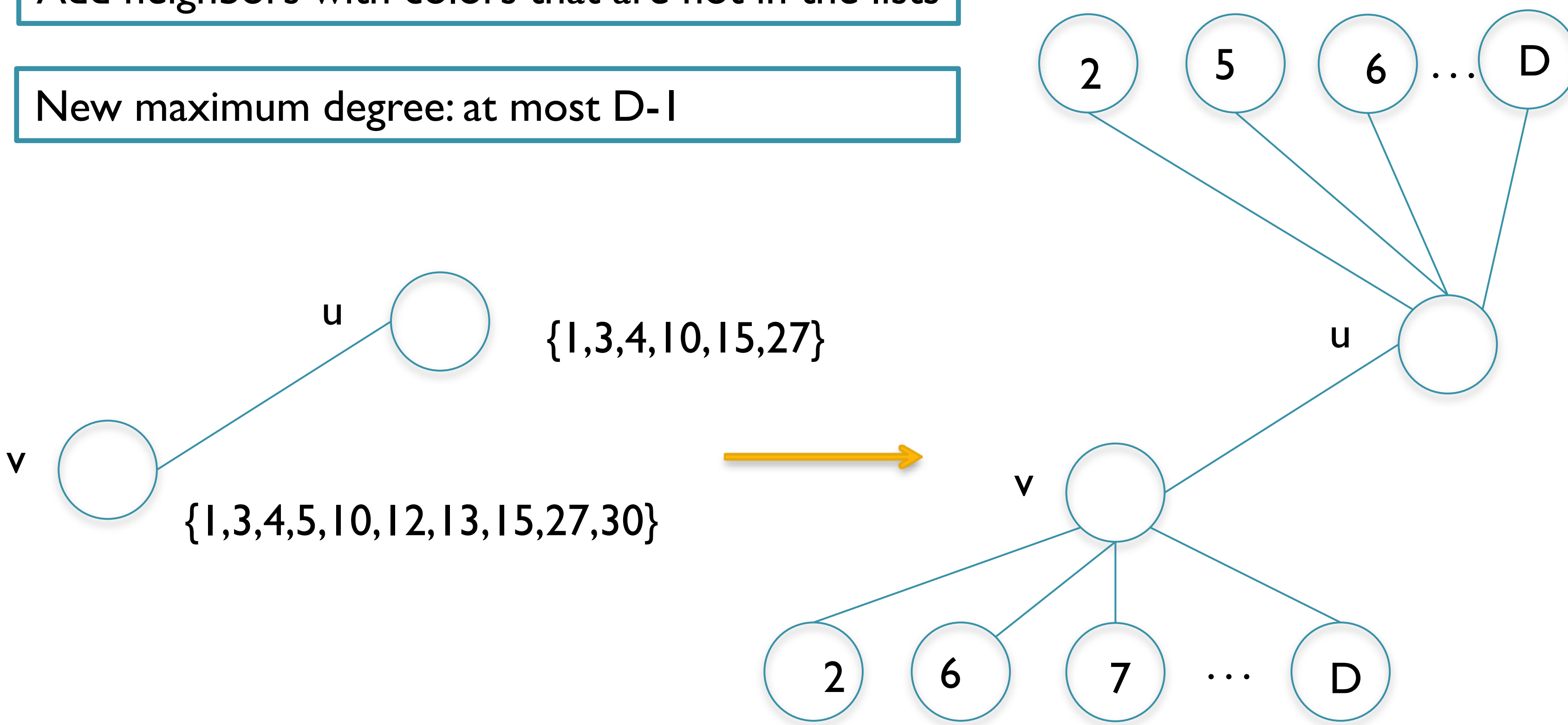


Static Algorithm for List-Coloring

Solution: a reduction from list coloring to coloring partially-colored graphs

Add neighbors with colors that are not in the lists

New maximum degree: at most D-1



Conclusion

- Static algorithms for graphs with partial solution yield dynamic algorithms.

- Static algorithms for graphs with partial solution are known for:
  - Coloring: $\sim O\left(\sqrt{\Delta} + \log^* n\right)$ time.
  - Maximal Independent Set: $O(\Delta + \log^* n)$ time.
  - Maximal Matching: $O(\Delta + \log^* n)$ time.
  - …

- We obtain **dynamic algorithms** for these problems with the same **update time**.

Can we do better than that?

Conclusion

- In these dynamic settings changes occur in steps.
- During an execution of an algorithm no changes occur.

Can algorithms cope with changes during their execution?

Thank you!