Random local algorithms from the graph limits perspective

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Why are random graphs interesting for us?

1. A property of random graphs means what these properties are true a.a.s. for every graphs, with the corresponding distribution.
2. Random graphs have extremal properties. (Better than the Petersen-graph.)
3. Graph limit theory is about separation of structure from randomness. To do that, we need to understand randomness.

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Beginner: matching ratio

Competent: independence ratio

Expert: chromatic number

Genius: homomorphism numbers

What is the **matching** ratio of random d-regular graphs?

(Size of the maximum matching divided by the number of vertices.)

Theorem. (Nguyen, Onak, 2008) $\exists$ a local algorithm computing an almost maximum matching (with εn error) on all graphs with degrees $\leqslant d$.

Corollary. Random d-regular graphs have an almost perfect matching.

Proof. There is a graph with a perfect matching which is locally (Benjamini–Schramm) equivalent to random d-regular graphs.

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Proof. # (3-regular graph on n vertices) $\gg$
(3-regular graph on n vertices ; independent set in it of size $0.46n$)

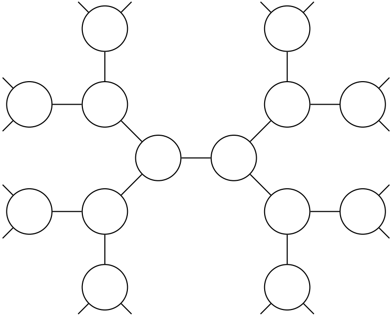
Corollary. No local algorithm can construct an almost maximum independent set. Not even on all 3-regular random graphs with large girth.

Proof. d-regular random graph and random bipartite graph are locally equivalent.

Local algorithms

= Constant-time distributed algorithms ≈ IID factor processes:

we assign a random seed to each vertex, and each vertex applies the same function on its constant-radius seeded neighborhood.



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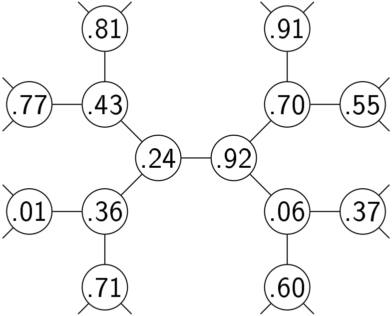

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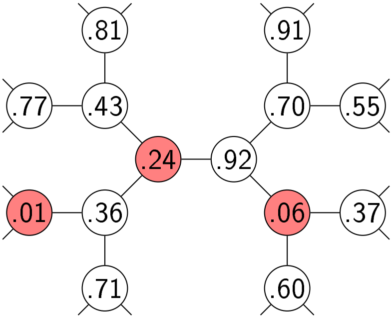

E.g. v outputs "yes" $\Longleftrightarrow \forall w \sim v: \ \mathrm{seed}(v) < \mathrm{seed}(w)$.
This constructs an independent set of expected size
$\sum_{v \in V} \frac{1}{1+\deg(v)} = \frac{n}{4}$ for 3-regular graphs.

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Upper bounds for independence ratio of 3-regular random graphs:

Bollobás, 1981:	$\alpha(3) < 0.4591$
McKay, 1987:	$\alpha(3) < 0.4554$
Lelarge, Oulamara, 2018:	$\alpha(3) < 0.45086$ stat. physics!
Balogh, Kostochka, Liu, 2019:	$\alpha(3) < 0.454$
(Cs, 2018++:	$\alpha(3) < 0.45087$)

Lower bounds only since 2010:

Kardoš, Kráł, Volec, based on Hoppen	$0.4352 < \alpha_{\text{local}}(3)$	(exact)
Cs, Gerencsér, Harangi, Virág:	$0.4361 < \alpha_{\text{local}}(3)$	(exact)
Cs, Gerencsér, Harangi, Virág:	$0.438 < \alpha_{\text{local}}(3)$	(stat)
Hoppen, Wormald	$0.4375 < \alpha_{\text{local}}(3)$	(exact)
Cs, based on Hoppen, Wormald:	$0.4453 < \alpha_{\text{local}}(3)$	(diff-eq)
Cs, Gerencsér:	$0.446 < \alpha_{\text{local}}(3)$	(stat)

To sum up:

$$0.446 < \alpha_{\text{local}}(3) \leqslant \alpha(3) < 0.451$$

A graph limit theory motivation

Structure: colored neighborhood distribution of the graph with a (vertex-)coloring. E.g. independent set, bisection, proper coloring, etc.

Question. (Hatami, Lovász, Szegedy, 2014) Do the d-regular random graphs have no more structure than what can be constructed by local algorithms? (Does the sequence of random d-regular graphs local-global converge to the Bernoulli-graphing of the d-regular tree?)

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(We will come back to it later.)

Bounds for random graphs and for local algorithms

$X(v)$: the output of the process at vertex v. Let degree $d = 3$.

Theorems. (Bowen, 2009; Rahman, Virág, 2017; Backhausz, Szegedy, 2018) Entropy inequalities including:

$$H\big(X(\circ\!-\!\circ)\big) \geqslant \frac{2d-2}{d} H\big(X(\circ)\big) = \frac{4}{3} H\big(X(\circ)\big)$$

Theorem. (Backhausz, Szegedy, Virág, 2015) If the outputs are real-valued, and $\mathrm{dist}(v,w) = r$, then

$$\big|\mathrm{corr}\big(X(v), X(w)\big)\big| \leqslant (r + 1 - \frac{2r}{d}) \cdot (d-1)^{-r/2} = \frac{1 + \frac{r}{3}}{\sqrt{2}^{r}}$$

Both inequalities are sharp and valid for random graphs, in some sense.

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Theorems. (Cs, Harangi, Virág: Entropy and Expansion, 2019+)
Generalizations of these inequalities for local algorithms:

- not just for trees (large-girth graphs, random graphs) but for (quasi-)transitive (regular) graphs, e.g. Cayley-graphs:

$$\mathrm{E}\Big(H(X(\circ\!-\!\circ))\Big) \geqslant \Big(1 + \frac{\text{edge-Cheeger}}{d}\Big) \cdot \mathrm{E}\Big(H(X(\circ))\Big)$$

- for a broader class of comparison sets (like edge vs. vertex)
- for other general uncertainty functions (like entropy and variance)
- where locality is generalized to other seed-vertex accessibility graphs

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Can we construct an almost maximum independent set for d-regular graphs by a local algorithm? – Results suggest that maybe yes for $d = 3$.

Why are we focusing on 3-regular graphs? – Because the problem is essentially the same for $d \geqslant 3$, and $d = 3$ is the easiest case.

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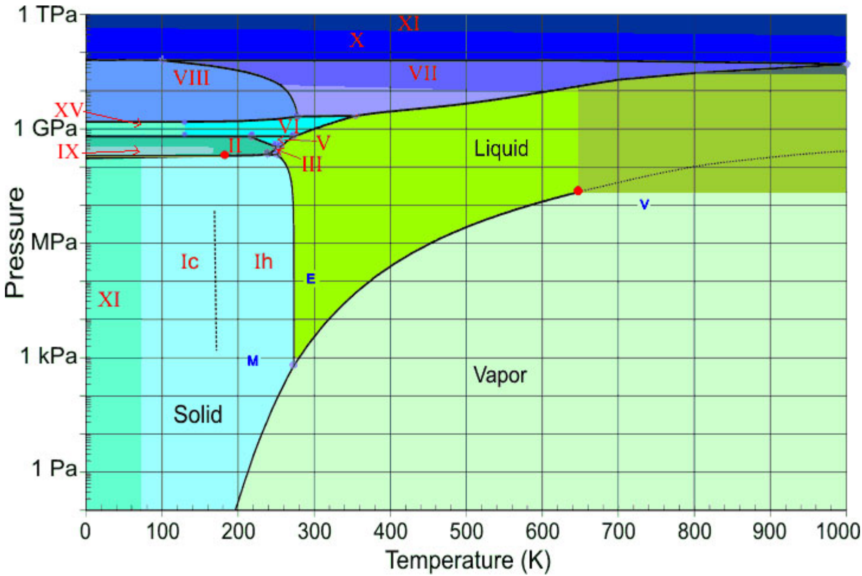
(But we did not know it before.)

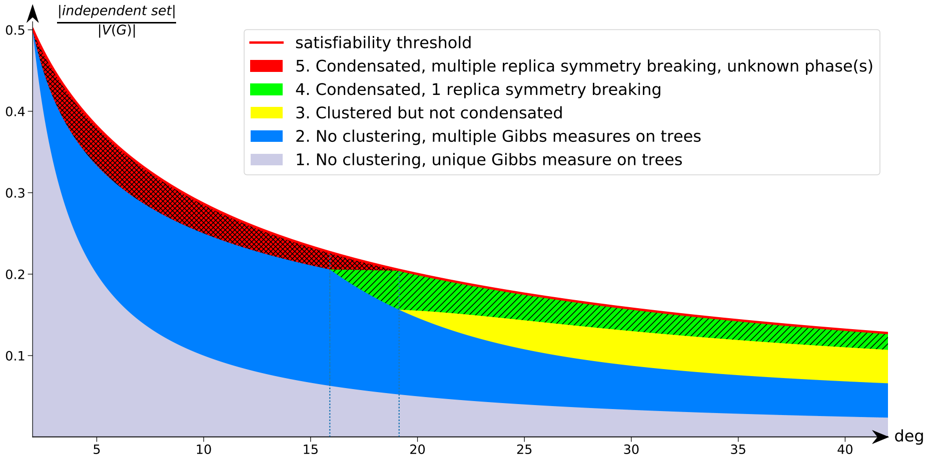
Theorem. (Gamarnik, Sudan, 2017, (Rahman, Virág)) For d large enough, the independence ratio of a random d-regular graph is $\frac{2\log(d)}{d}$, while local algorithms can find only $\frac{\log(d)}{d}$. (Multiplied by $1 + o_d(1)$.)

But this implies really nothing for small degree d.

Phase transitions: If we want to understand the structure of water, we need to know that it has different phases (ice, water, vapor, etc.) depending on pressure and temperature. (The 19th solid state of water was just discovered...)



The phase diagram of independent sets, based on theorems, conjectures and best guesses



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Leaves open:

- Are random graphs the least structured d-regular large-girth graphs?
- Are there locally unconstructible structures which are present in all / almost all d-regular large-girth graphs?
- Are polynomial-time algorithms more efficient than local algorithms?
- How do these depend on the degree d?

Ongoing research with Ferenc Bencs and Viktor Harangi

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Proof. If # (d-regular graph on n vertices) $\gg$ # (d-regular graph on n vertices ; independent set in it of size $\alpha \cdot n$), then $\alpha(d) < \alpha$.

Equivalent forms of the proof. For typical processes we have:

$$H\big(X(\circ\!-\!\circ)\big) \geqslant \tfrac{2d-2}{d} H\big(X(\circ)\big) \qquad \text{and} \qquad H\big(X(d\text{-star})\big) \geqslant \tfrac{d}{2} H\big(X(\circ\!-\!\circ)\big)$$

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$$\mathrm{E}_G \left| \left\{ (V^*, E^*) \text{ is a cluster in } G: \ |V^*| + |E^*| = \alpha \cdot n \right\} \right|$$

$$\approx \sum_{\Delta(X(*))} e^{n\left(H(X(*)) - \frac{d}{2} H(X(\circ-\circ))\right)} \approx \sup_{\Delta(X(*))} e^{n\left(H(X(*)) - \frac{d}{2} H(X(\circ-\circ))\right)}$$