Distance Computation in Massive Graphs

Michal Dory, University of Haifa

Computing Distances

Big Data

How to do a computation when the input is too large and cannot be stored in one machine?

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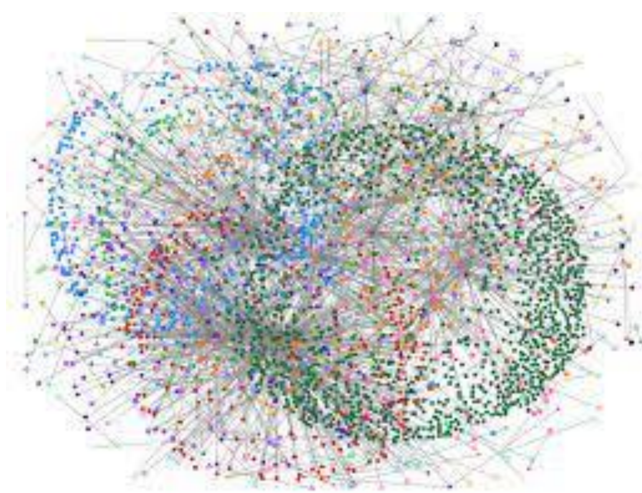
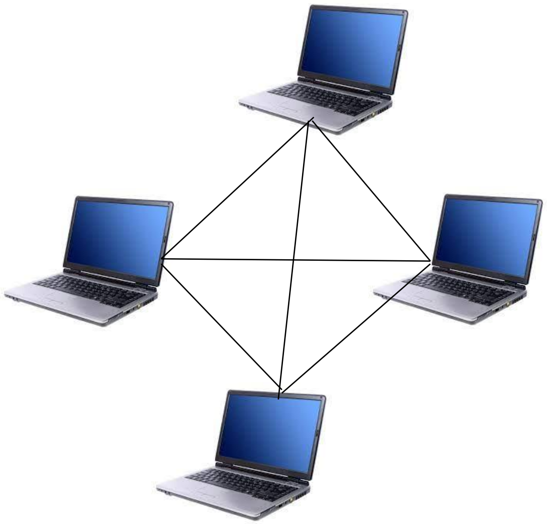
Development of practical systems for massively parallel computation: MapReduce, Hadoop, Spark, Dryad

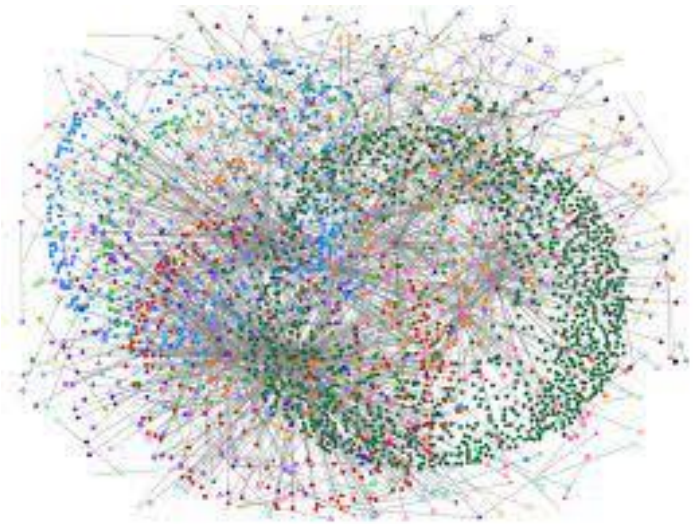


This Talk

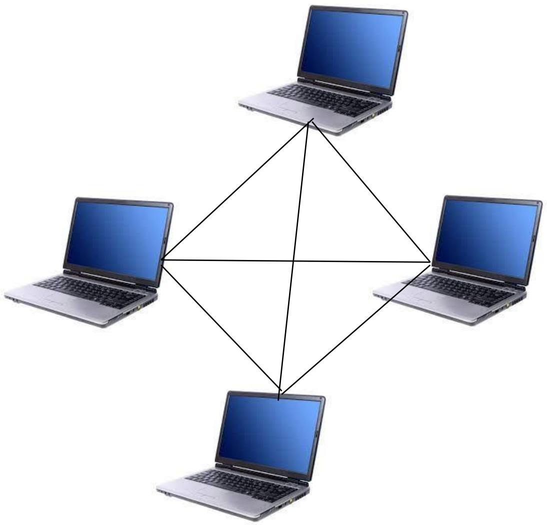
<u>Computing distances:</u>

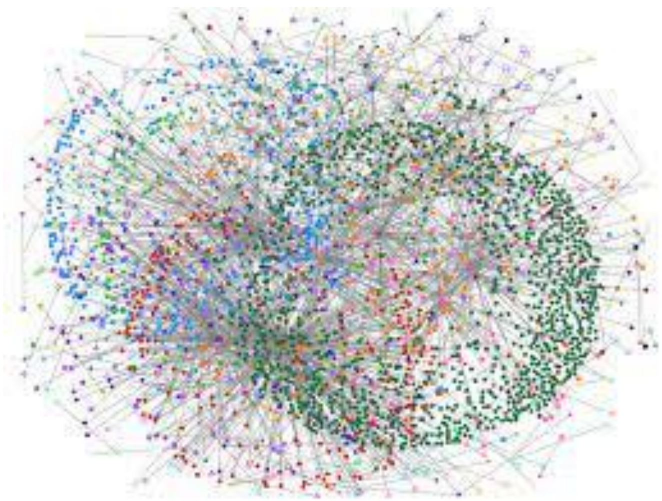
[Censor-Hillel, D, Korhonen, Leitersdorf, 2019]
[D, Parter, 2020]
[D, Fischer, Khoury, Leitersdorf, 2021]
[Biswas, D, Ghaffari, Mitrovic, Nazari, 2021]





The Model

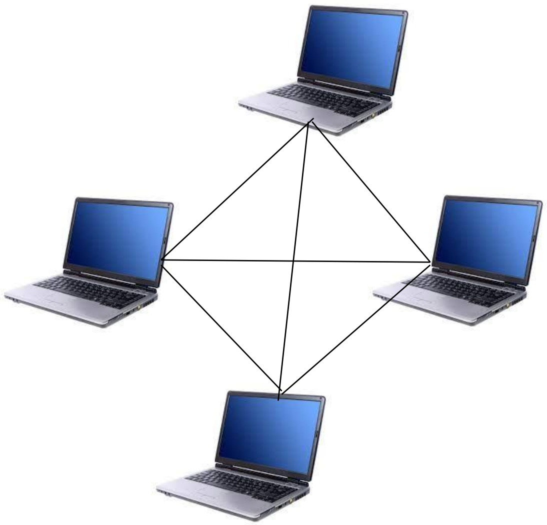


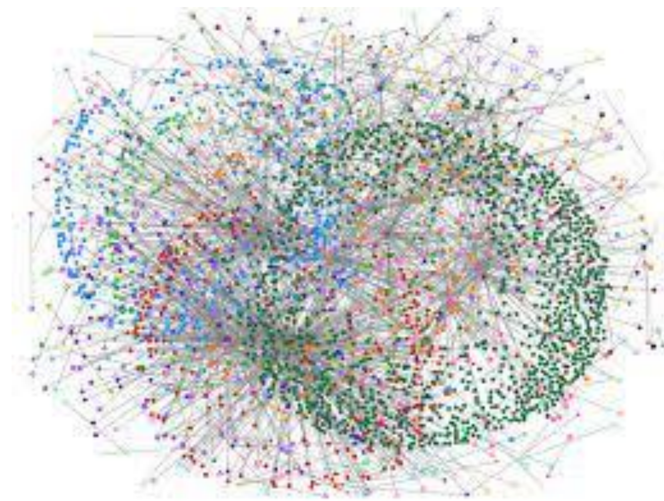


- n machines

n vertices

The Model





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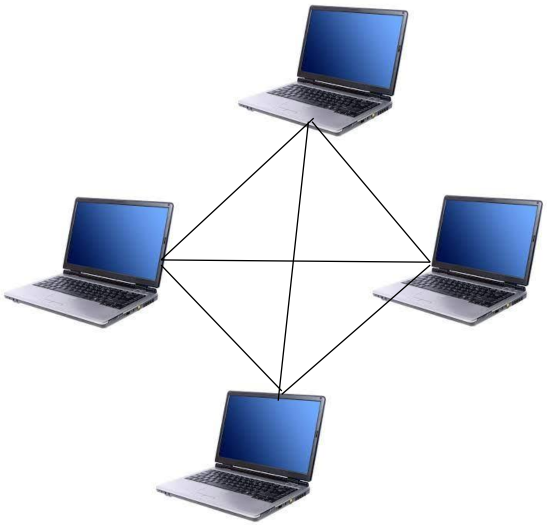
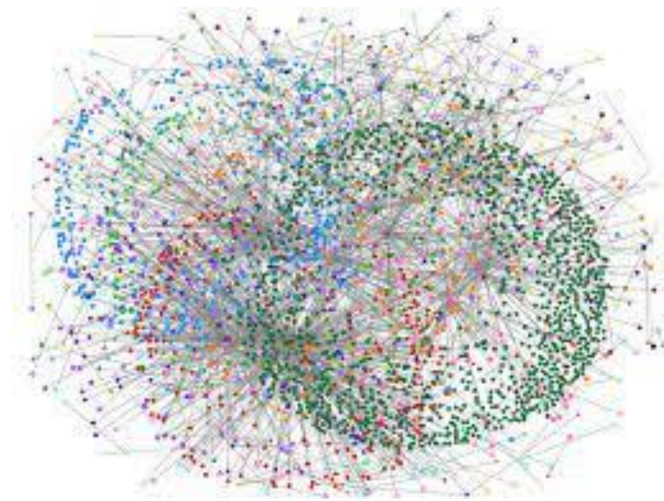
- Input per machine: $O(n)$ edges

The Model





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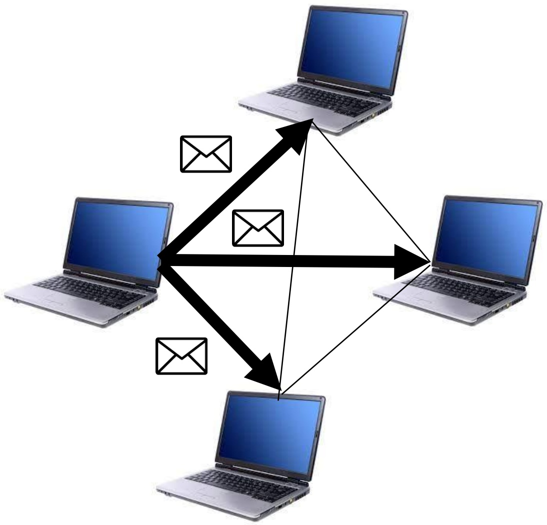

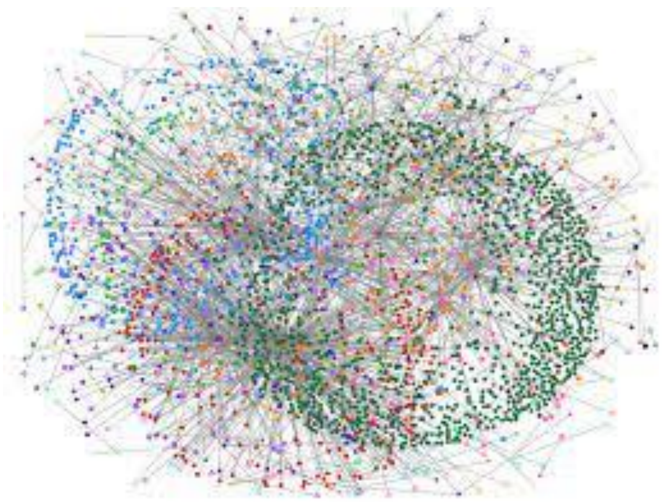

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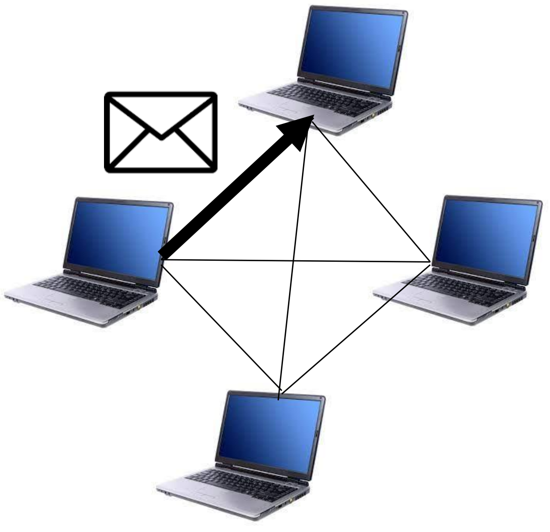

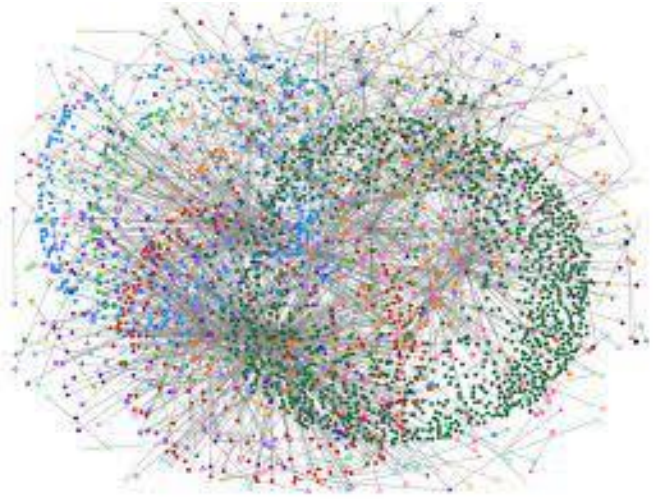

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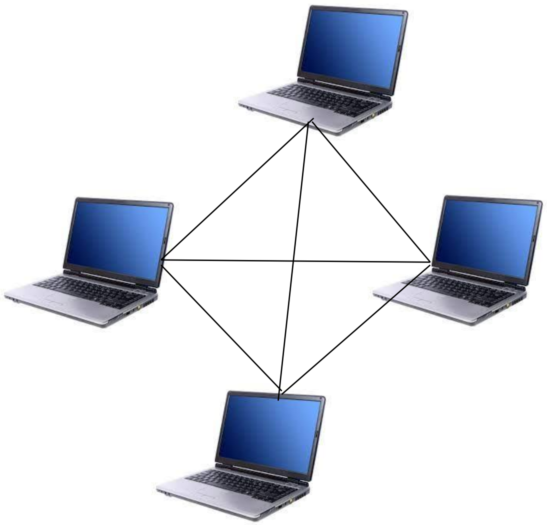
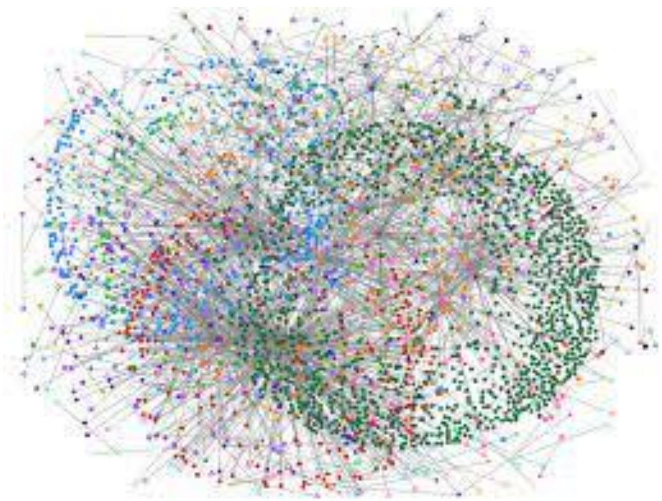
The Model - Congested Clique





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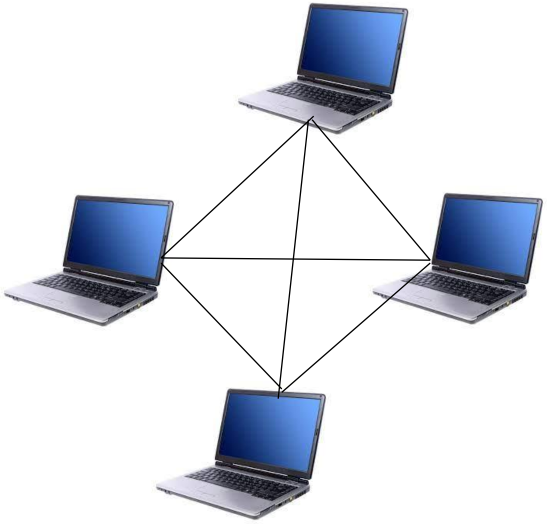
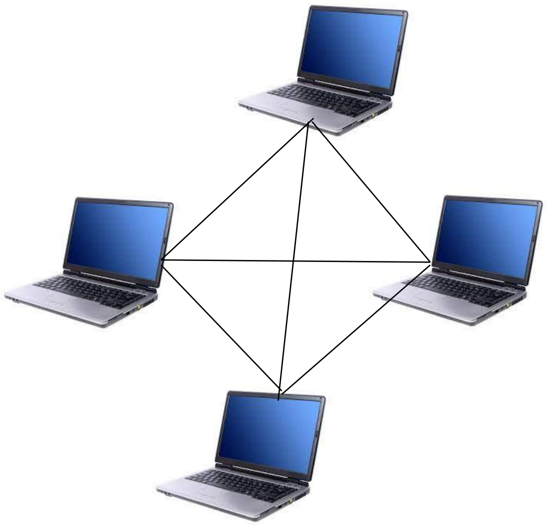
In the AMG workshop (Friday)

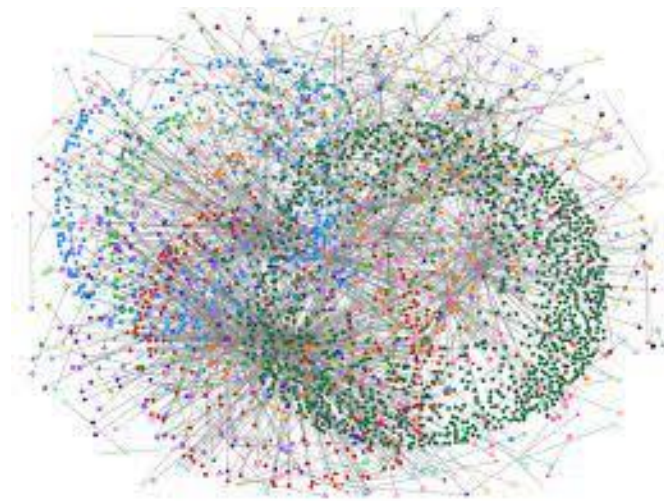
Yasamin Nazari – Distance Computation in MPC and related models



Machines have sublinear memory

The Model - Congested Clique





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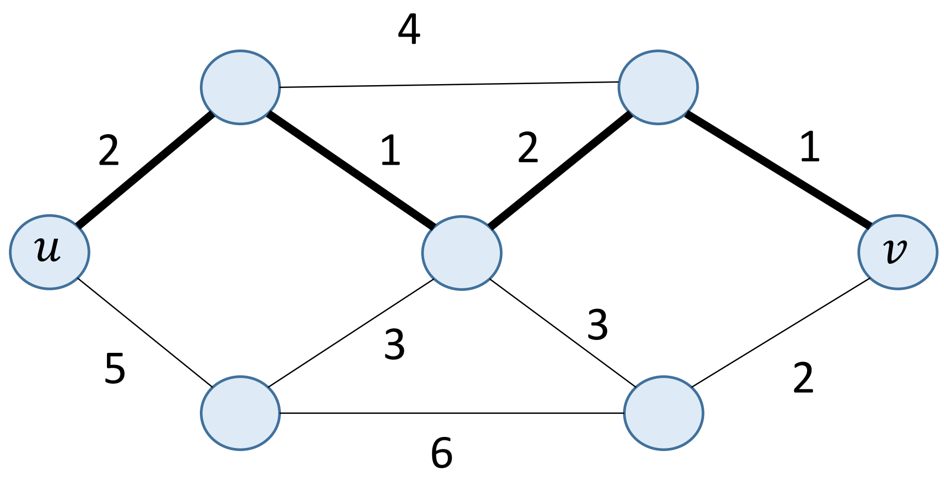
Distance Computation

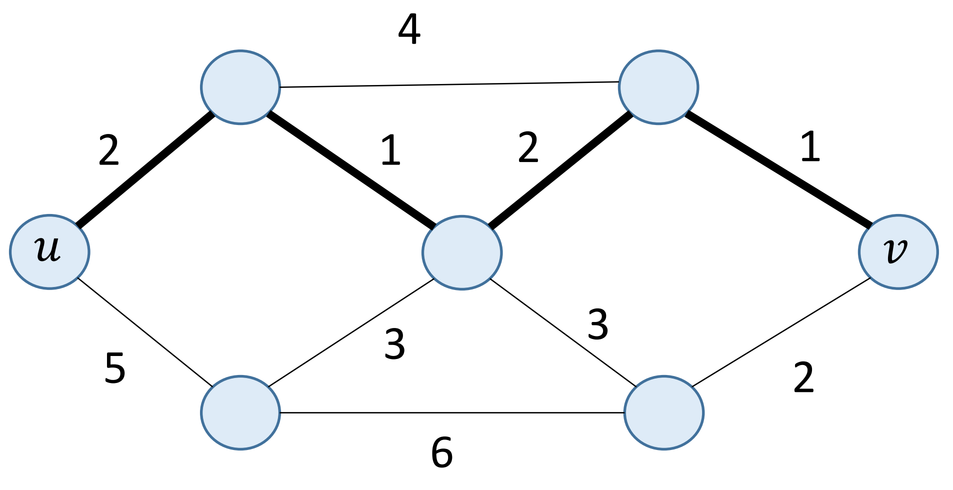
All-pairs shortest paths (APSP)



Distance Computation

All-pairs shortest paths (APSP)

<u>Focus</u>: APSP in unweighted undirected graphs

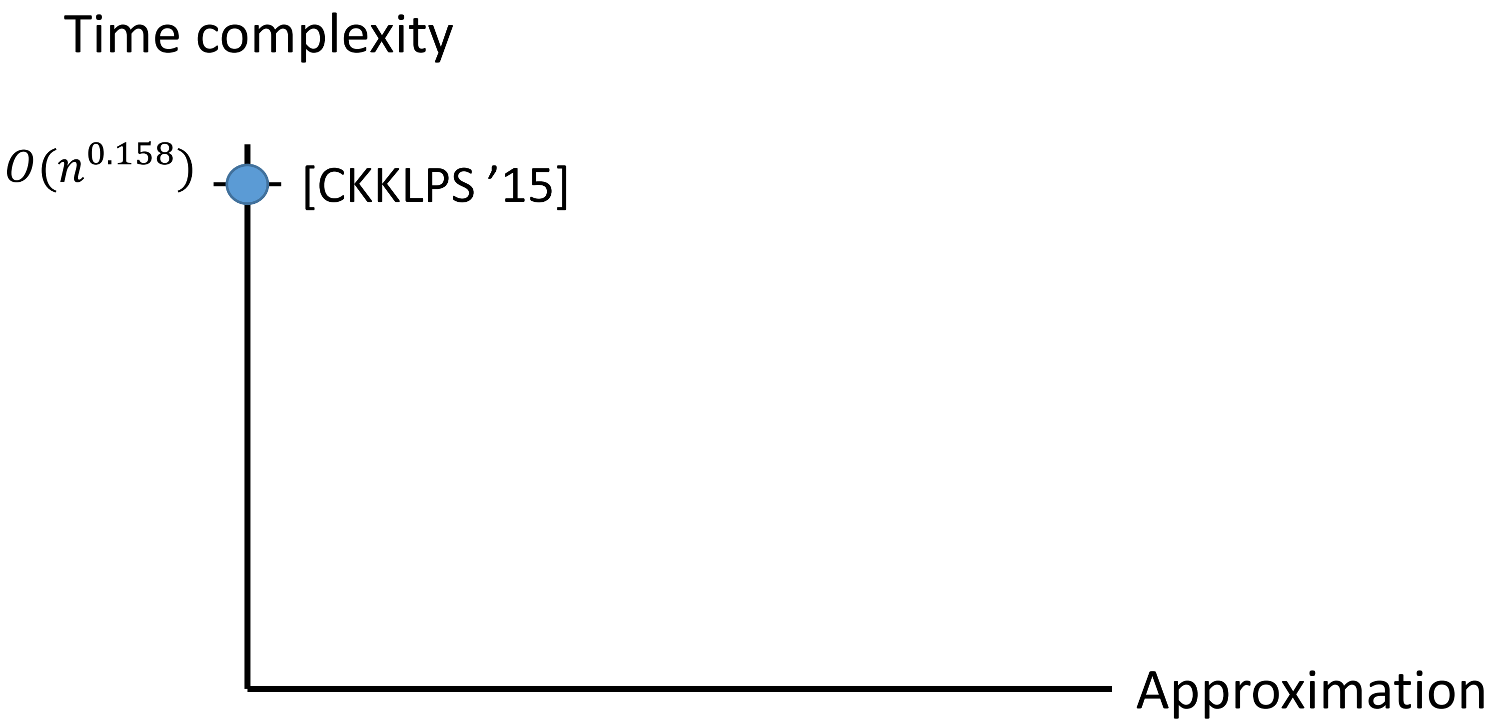


Distributed APSP

- Polynomial time algorithms for exact APSP based on matrix multiplication [Censor-Hillel et al. 15, Le Gall 16]

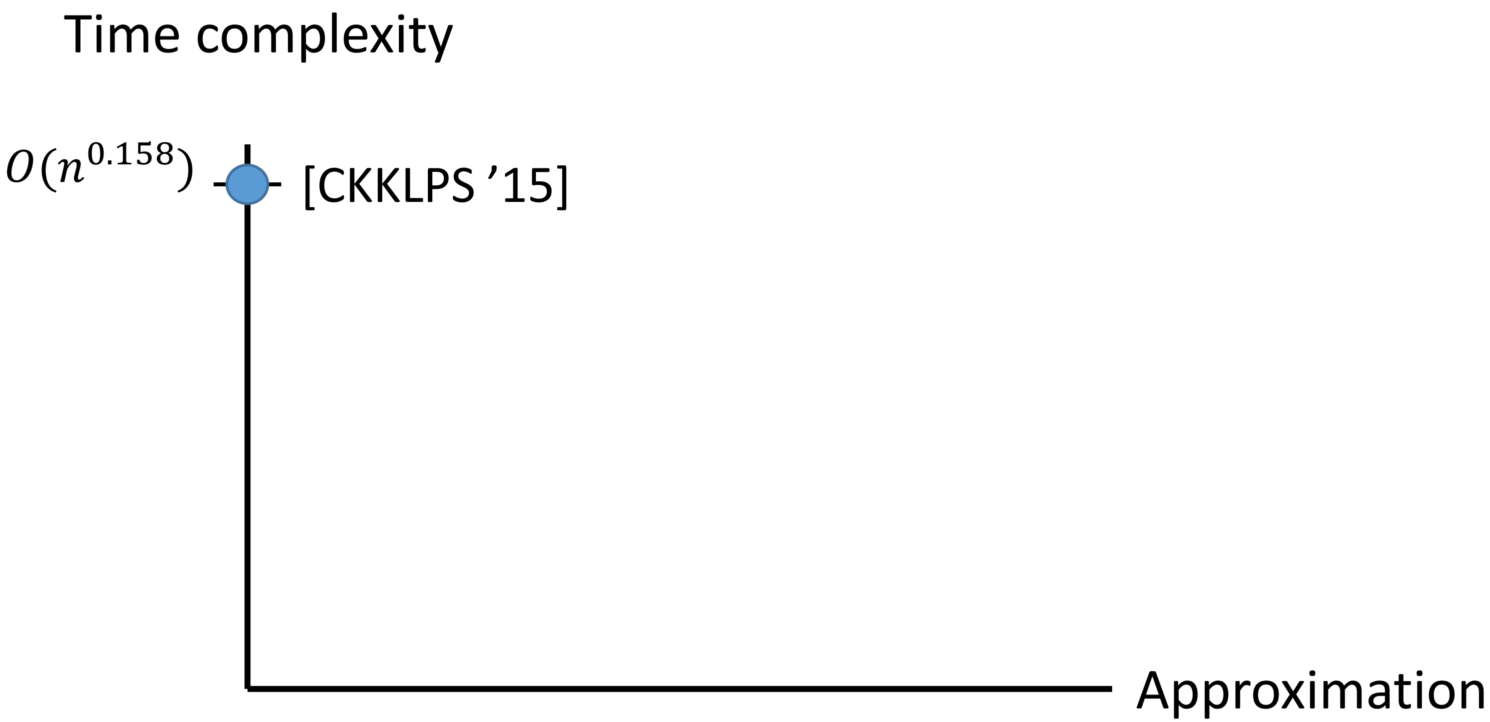
Round Complexity	Variant
$\tilde{O}\left(n^{1/3}\right)$	weighted directed
$O\left(n^{0.158}\right)$	unweighted undirected

Distributed APSP



Distributed APSP

Can we get faster algorithms if we allow approximations?



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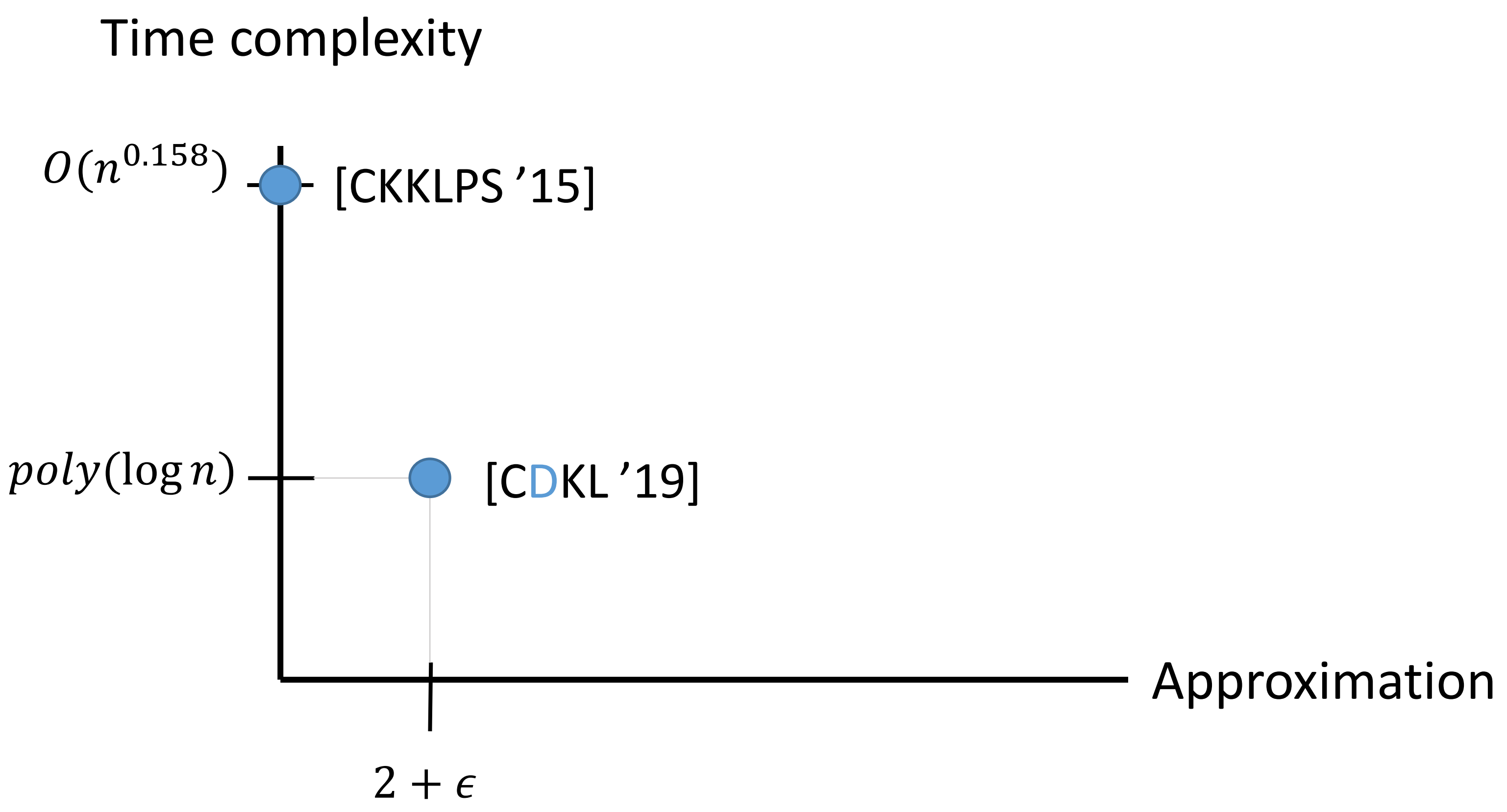

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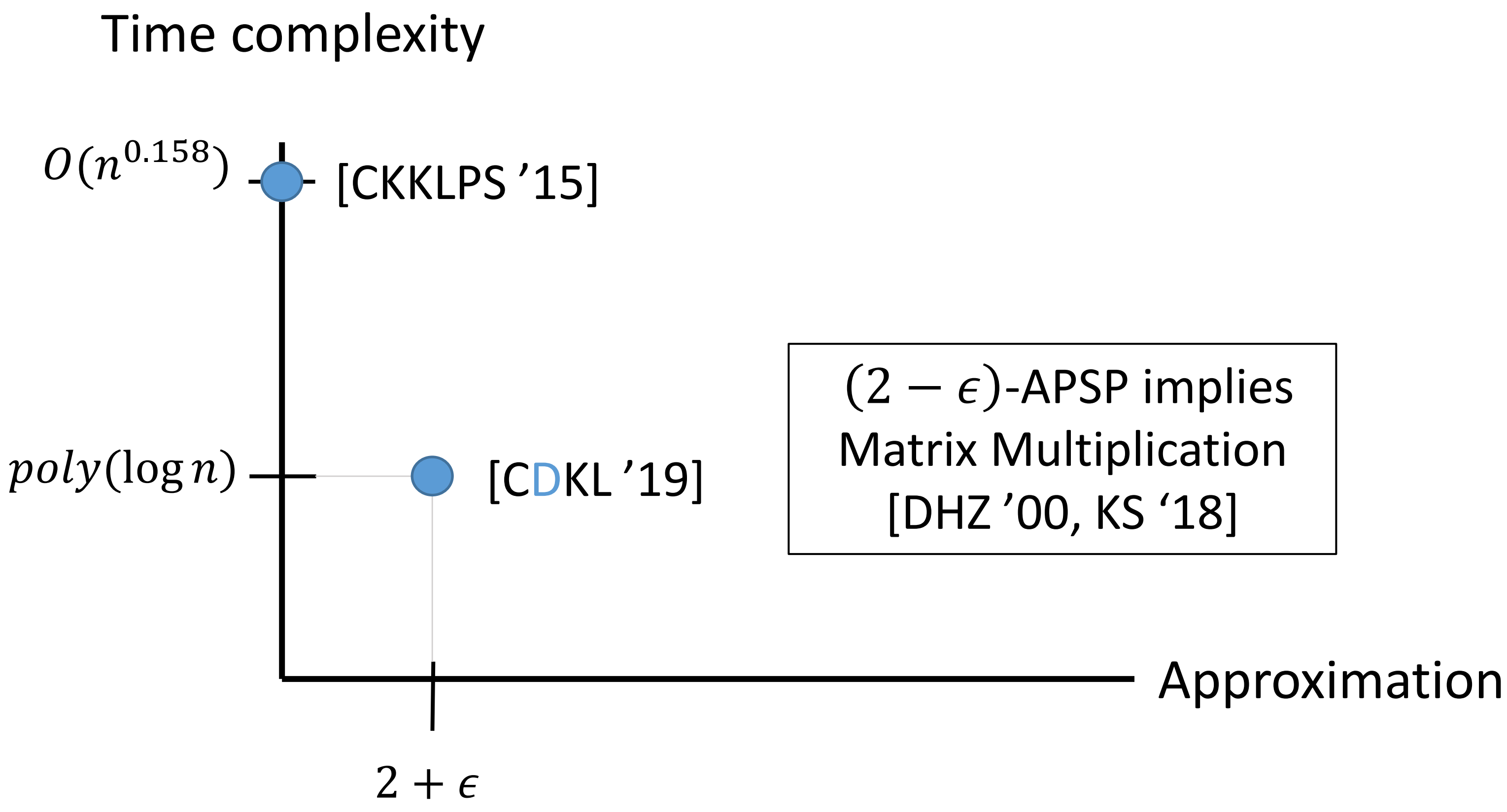

$(2-\epsilon)$-APSP implies Matrix Multiplication [DHZ '00, KS '18]

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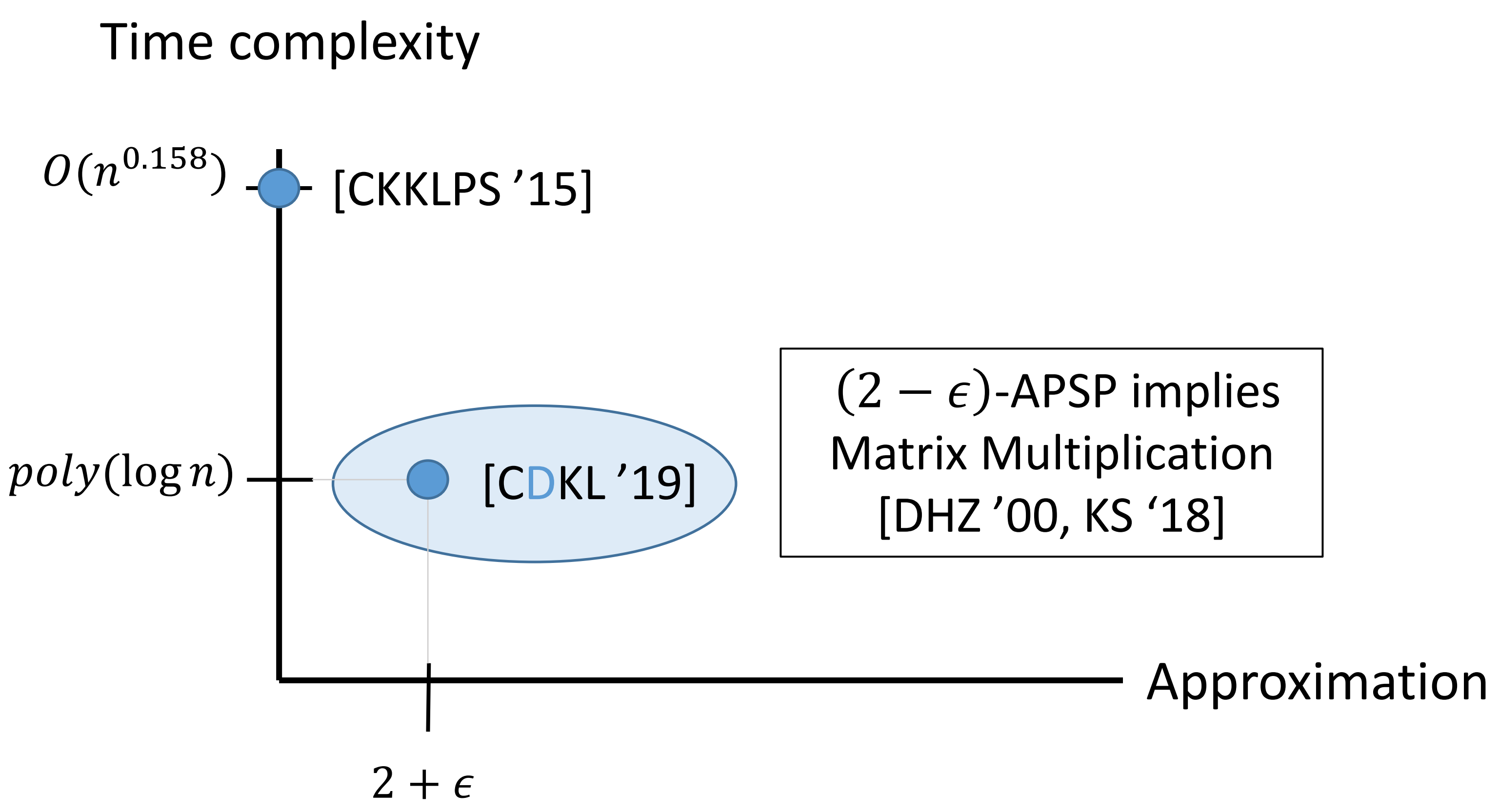


Distributed APSP

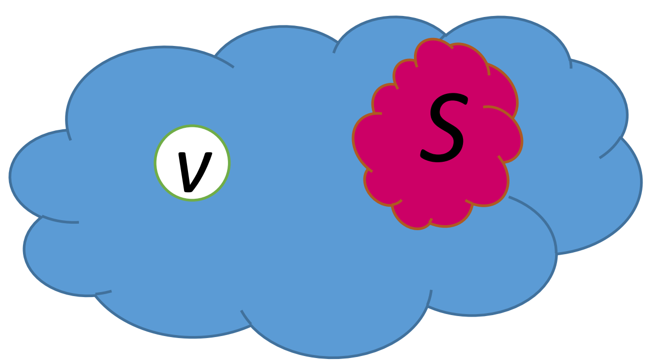
Matrix Multiplication $\Rightarrow$ Exact APSP

Sparse Matrix Multiplication $\Rightarrow$ $(2+\epsilon)$-APSP

Distributed APSP

Matrix Multiplication → Exact APSP

Sparse Matrix Multiplication → Distance Tools → $(2+\epsilon)$-APSP



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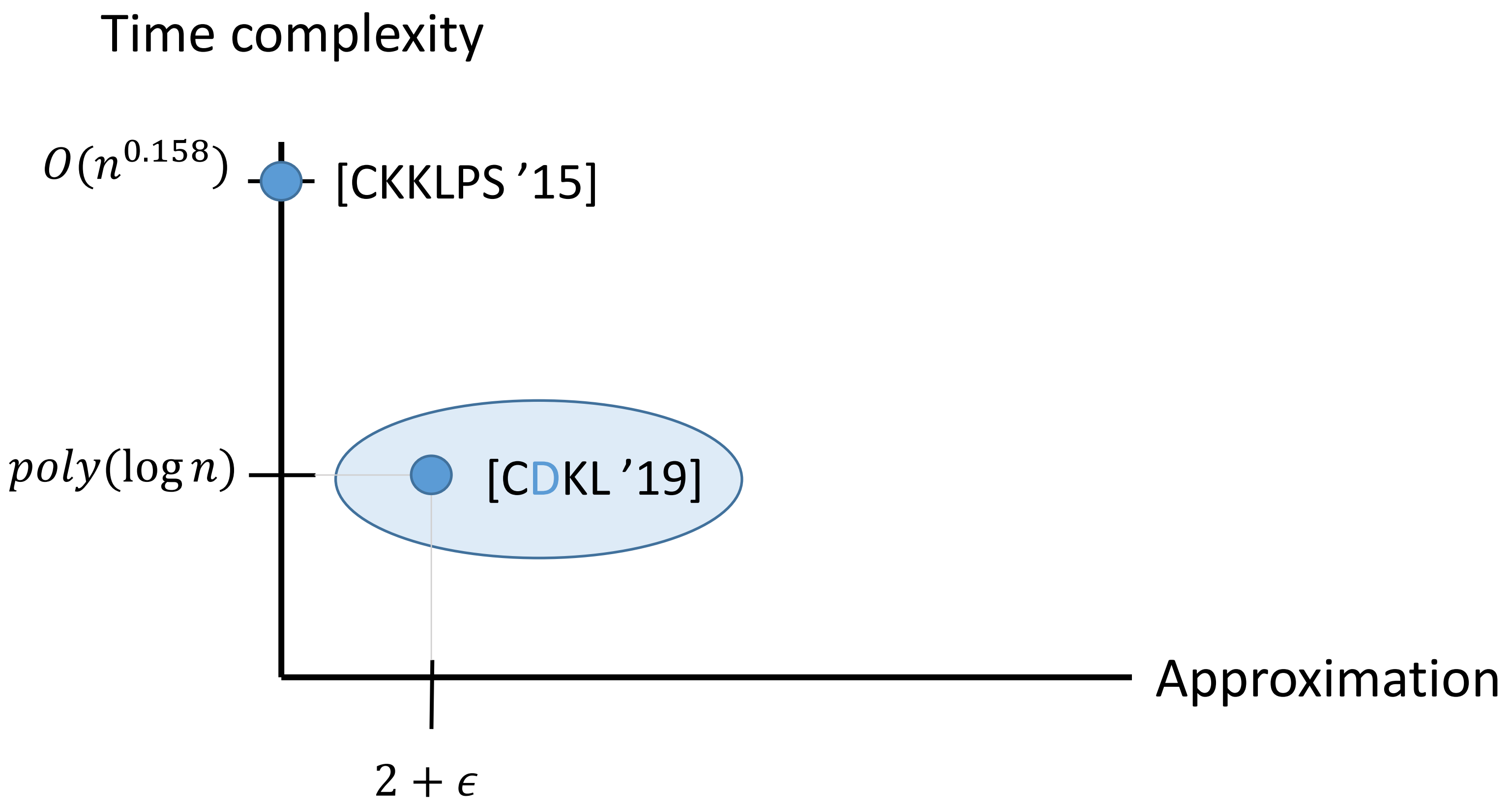

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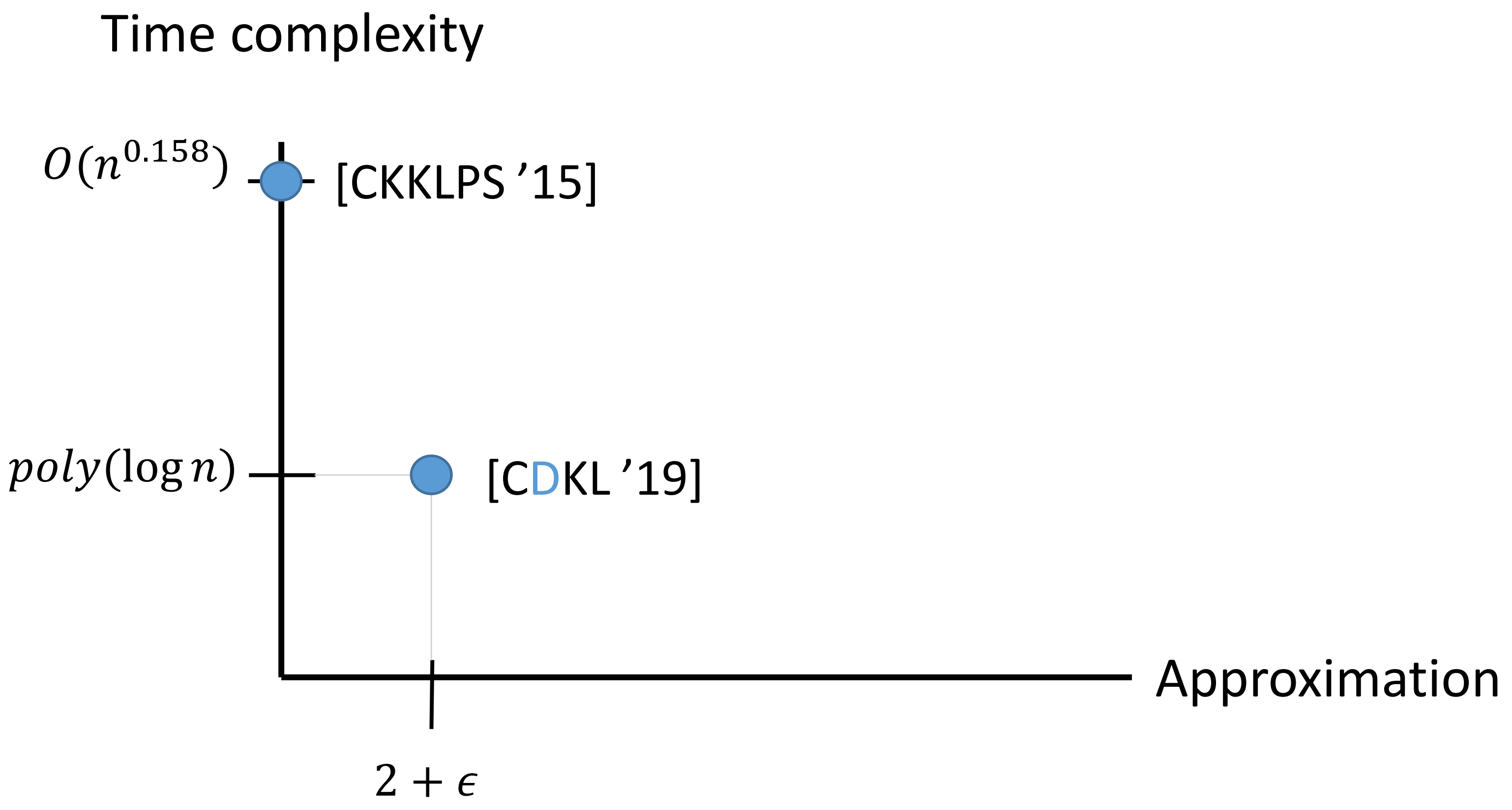

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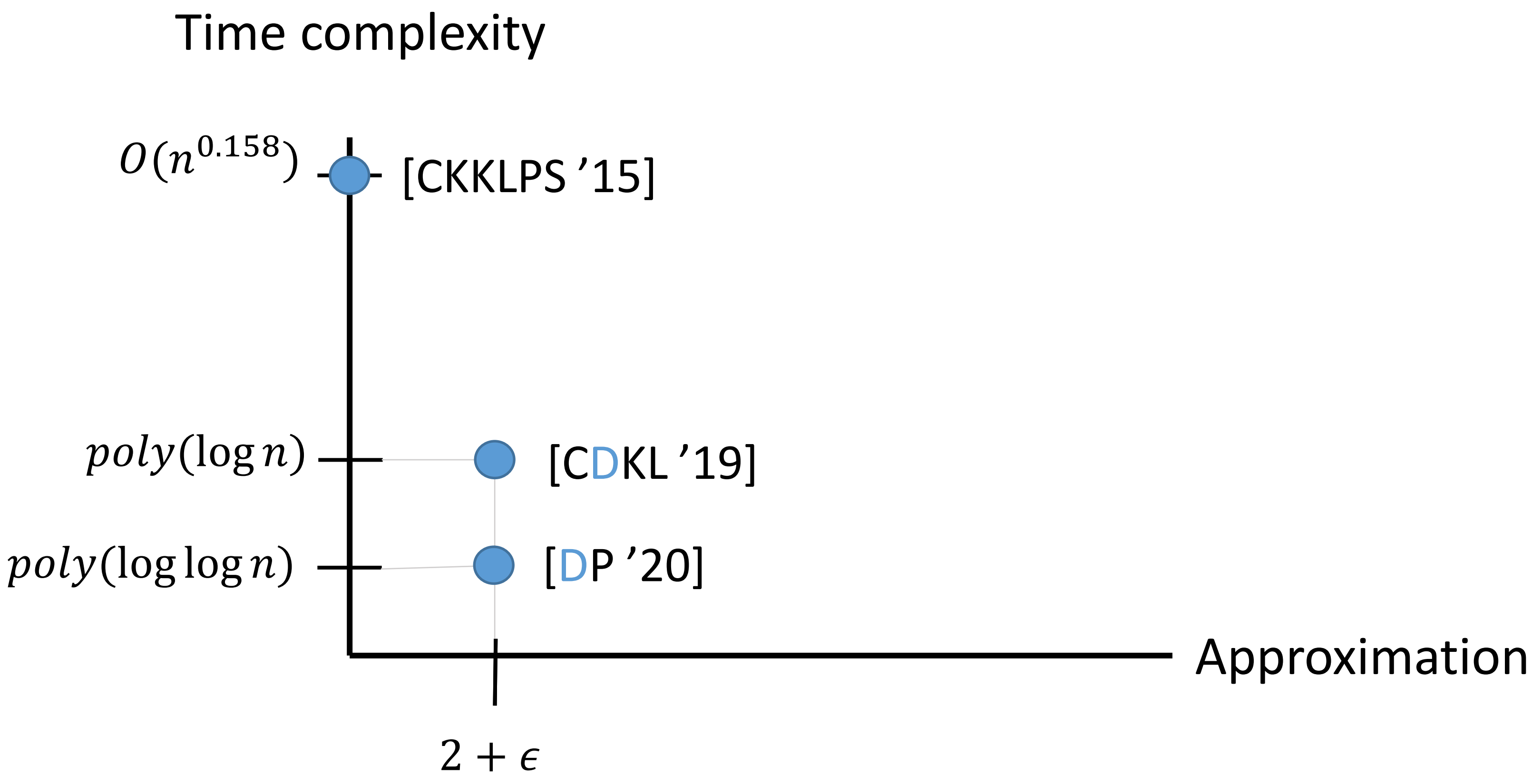

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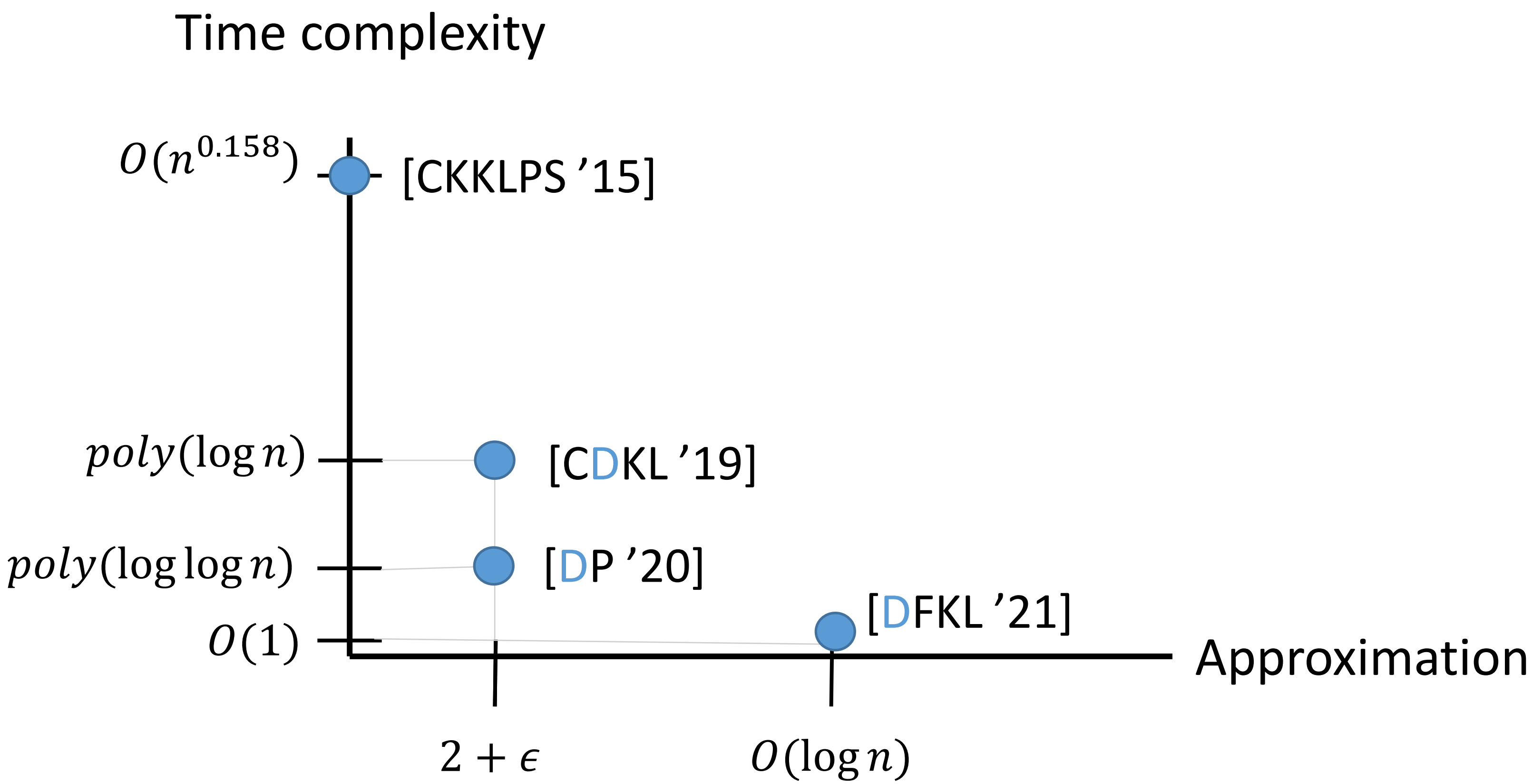

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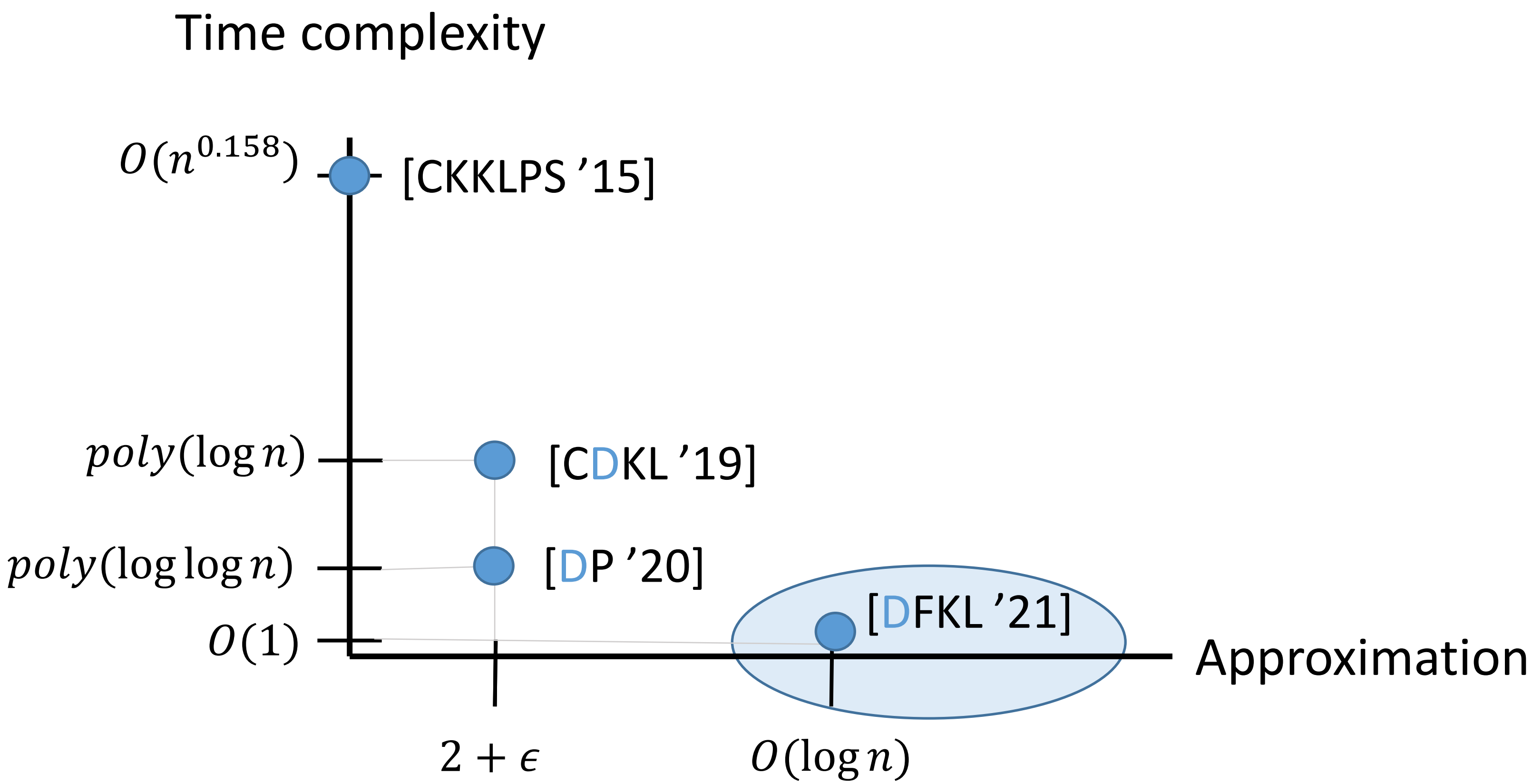
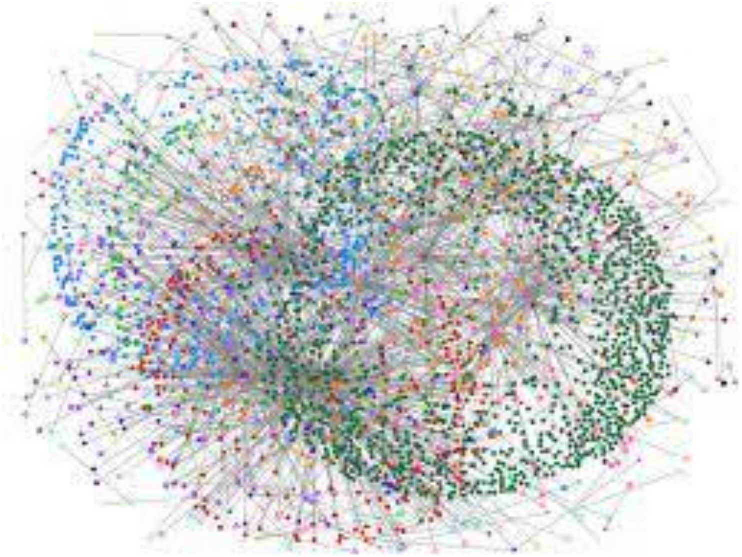


Our Techniques

Goal: compress our graph while preserving the distances



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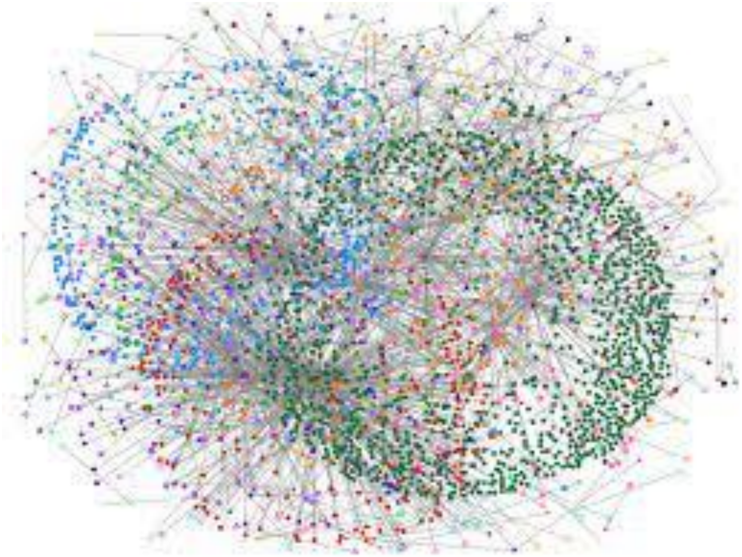
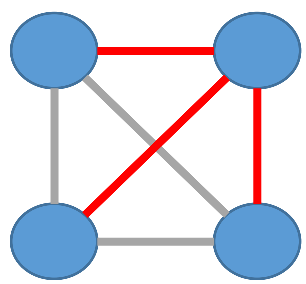


Compute a spanner, a sparse subgraph that approximately preserves the distances

Spanners

- A ***k*-spanner** of a graph G, is a subgraph H of G such that for all u, v:

$$d_G(u, v) \leq d_H(u, v) \leq k \cdot d_G(u, v)$$



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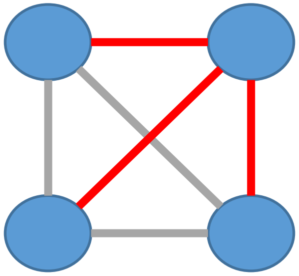
<u>Numerous applications</u>:

- Synchronization in distributed networks
- Compact routing schemes
- Approximate shortest paths
- …



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For $k = \log n$: $O(\log n)$-spanner of size $O(n)$

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$\Downarrow$

$O(\log n)$-approximation for APSP in $O(1)$ rounds

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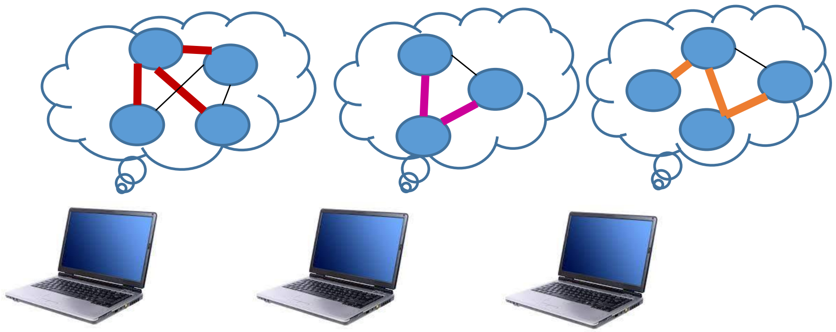
$O(\log n)$-approximation for APSP in $O(1)$ rounds

The Spanner Algorithm

Edge Sparsification



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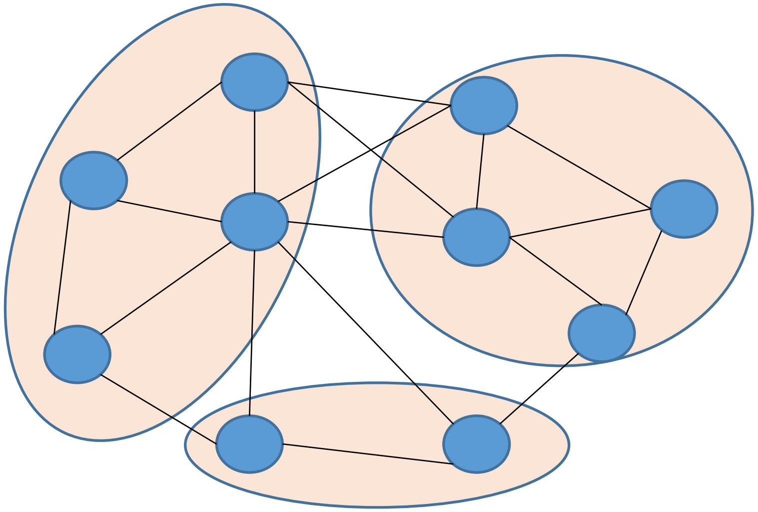

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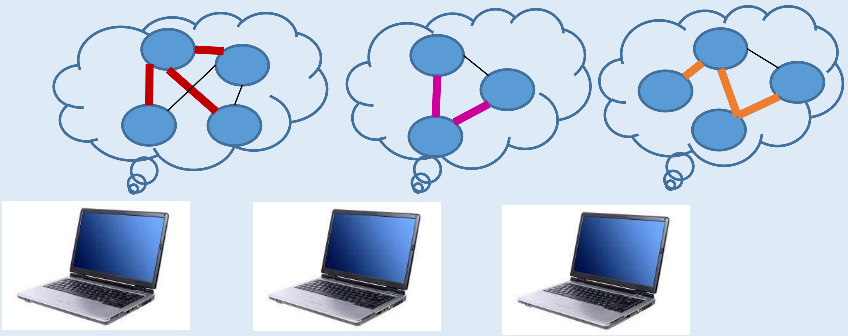

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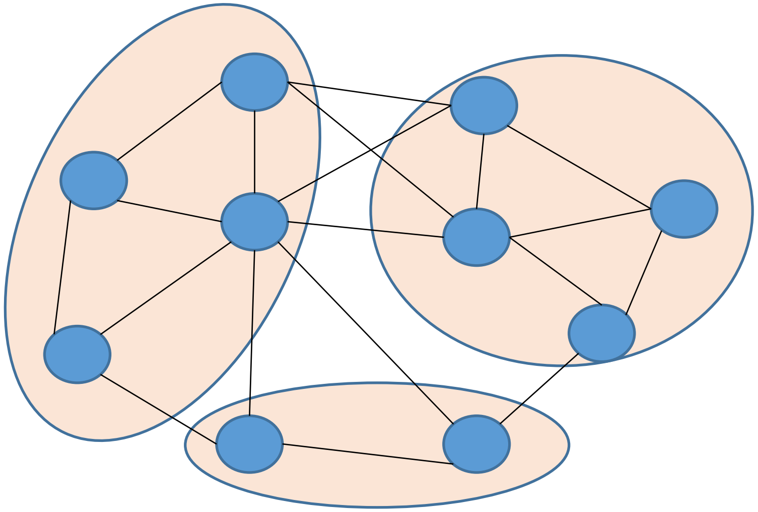
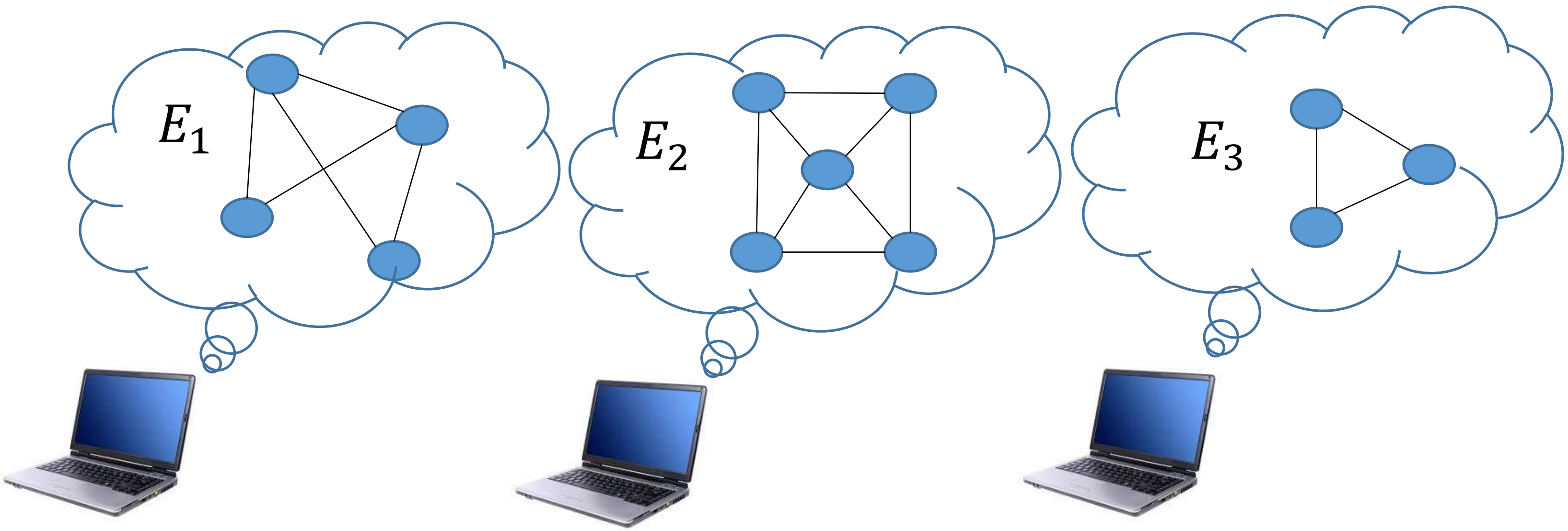


Edge Sparsification

Partition the edges: $E = \cup E_i$



Compute a spanner on each part separately

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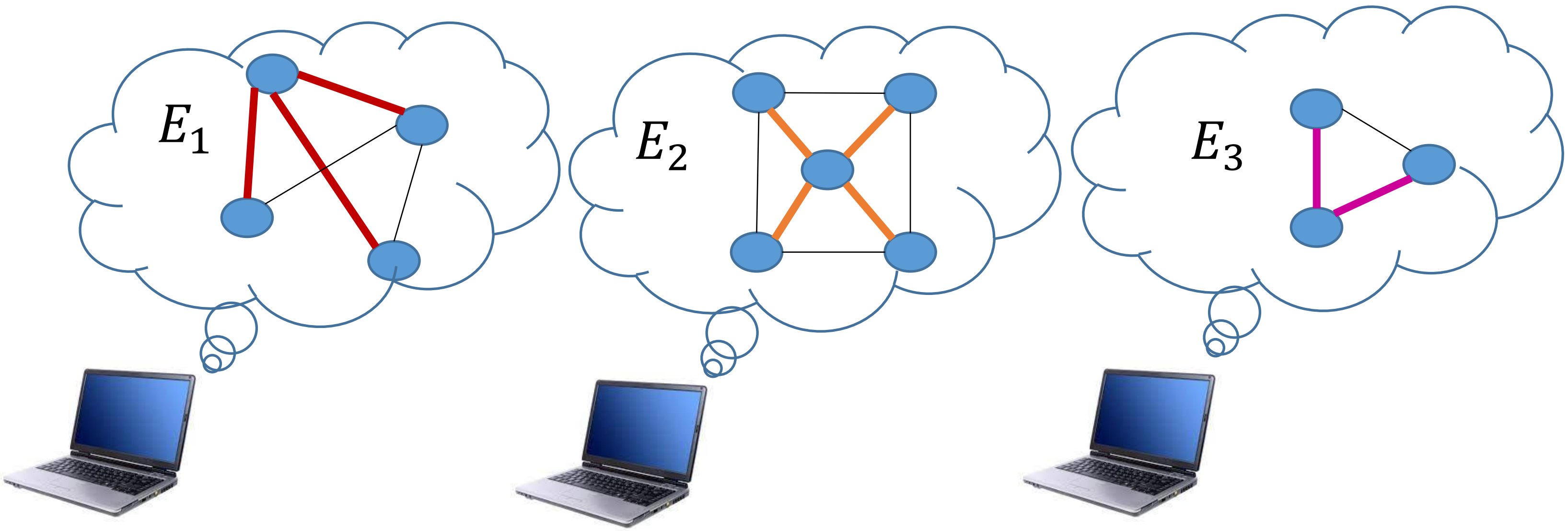

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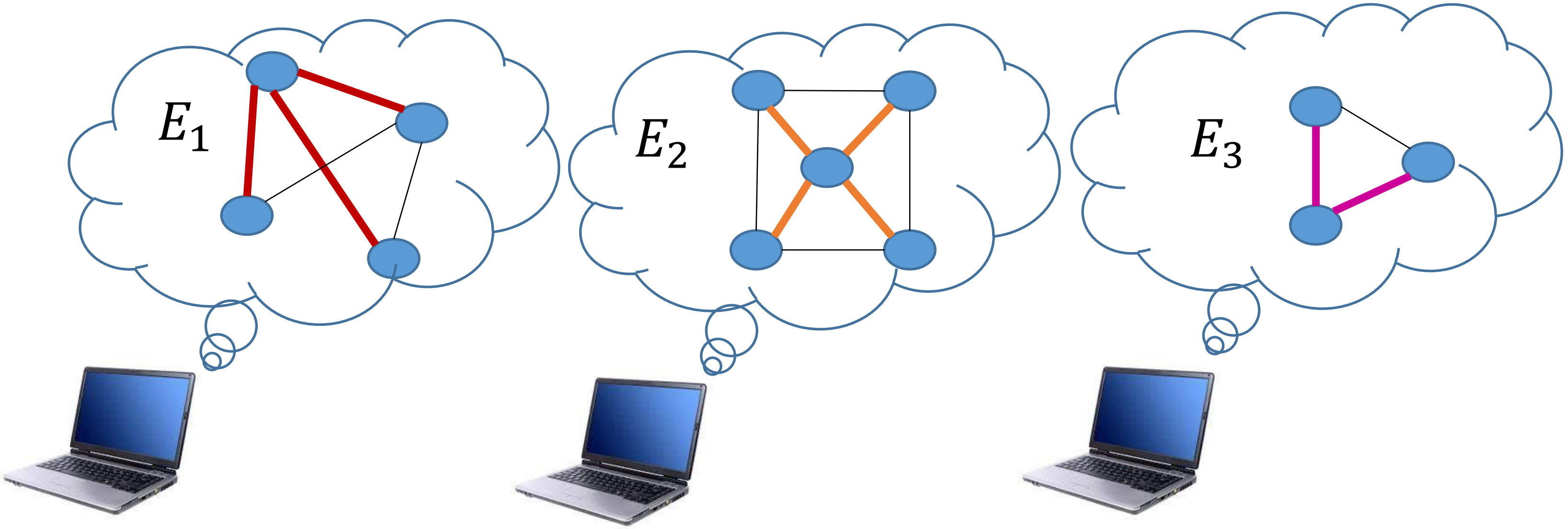

<u>Claim:</u> the union of the spanners is a spanner of the original graph

Edge Sparsification

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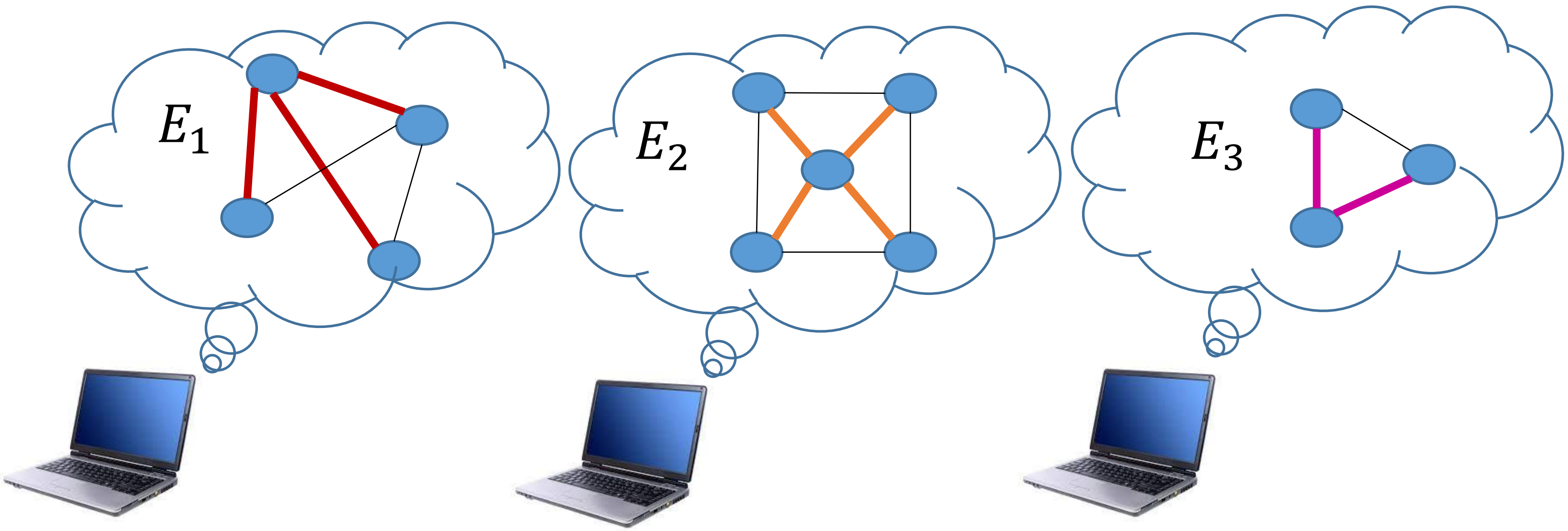

Problem: what is the size of the spanner?

Edge Sparsification

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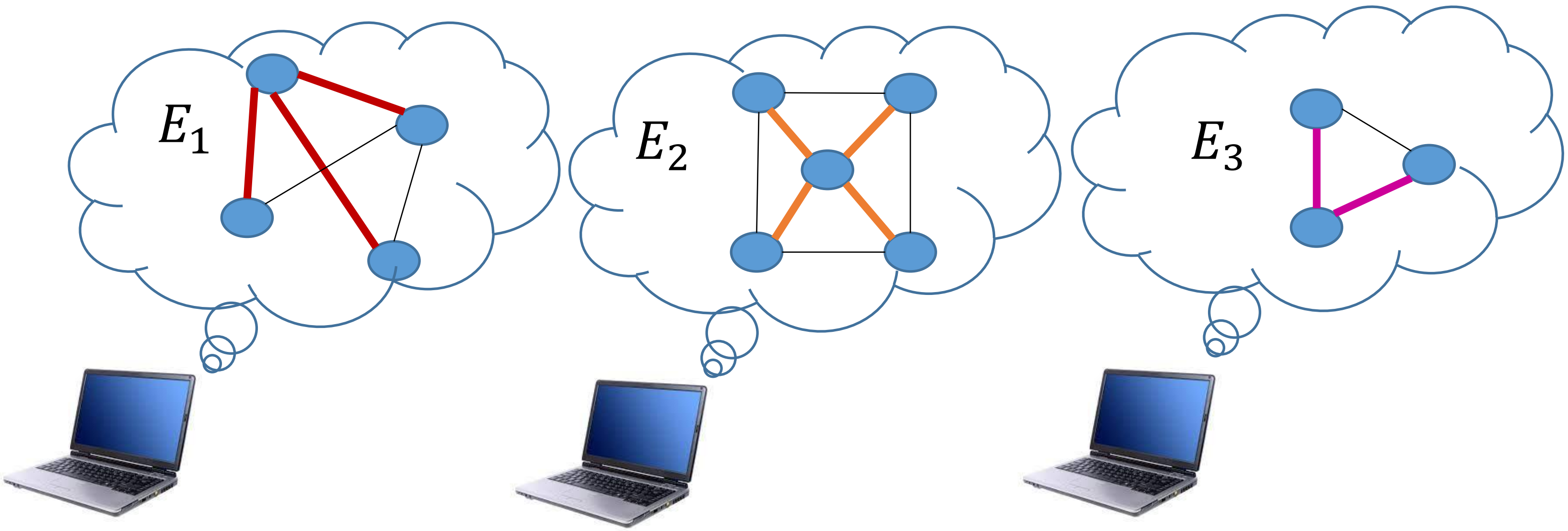
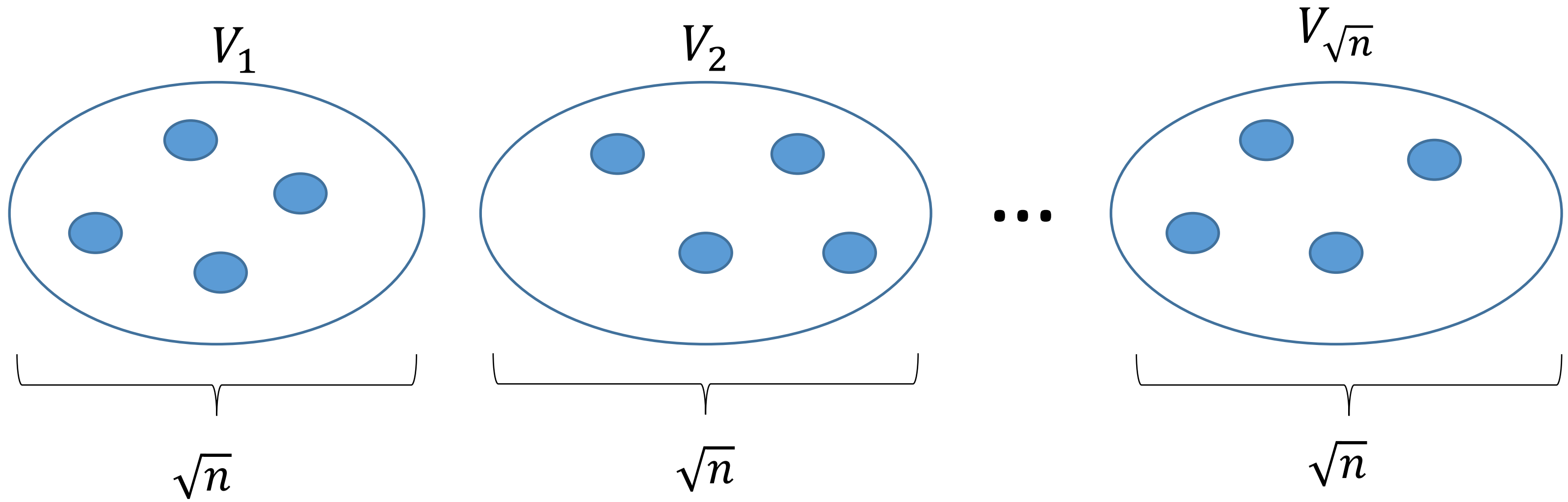


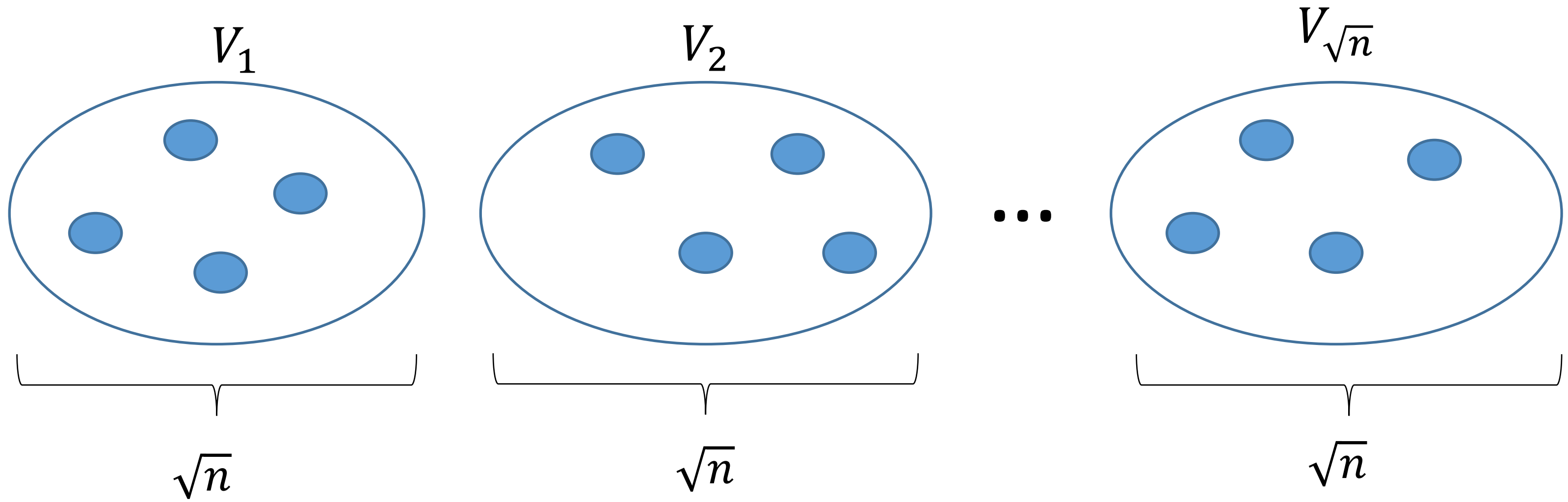
Challenge: find a smart partitioning

Edge Partitioning: Try 1



Partition the vertices into $\sqrt{n}$ sets of size $\sqrt{n}$

Edge Partitioning: Try 1



$$E_{i,j} = \left\{\{u, v\} \in E \mid u \in V_i, v \in V_j\right\}$$

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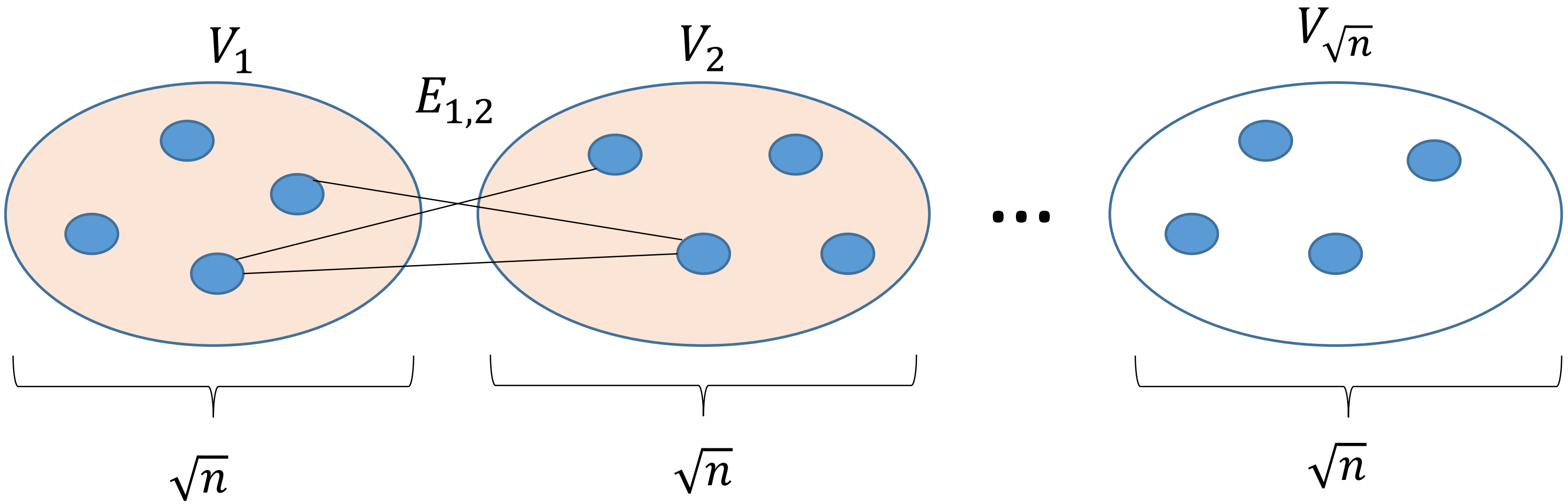

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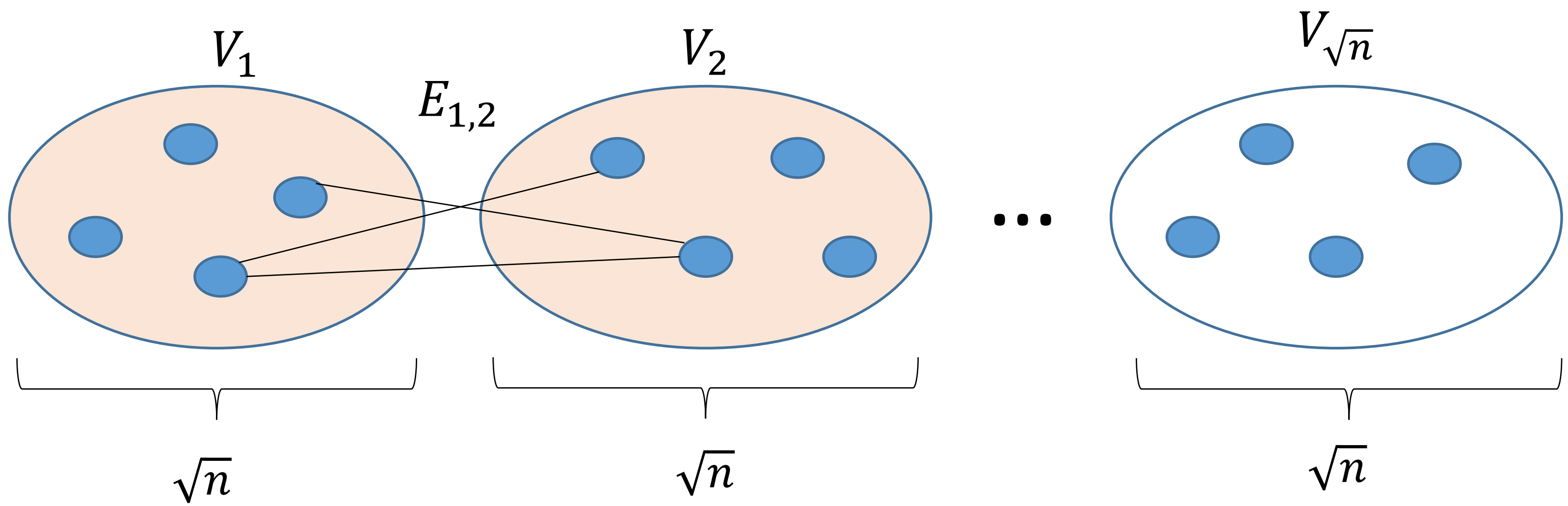

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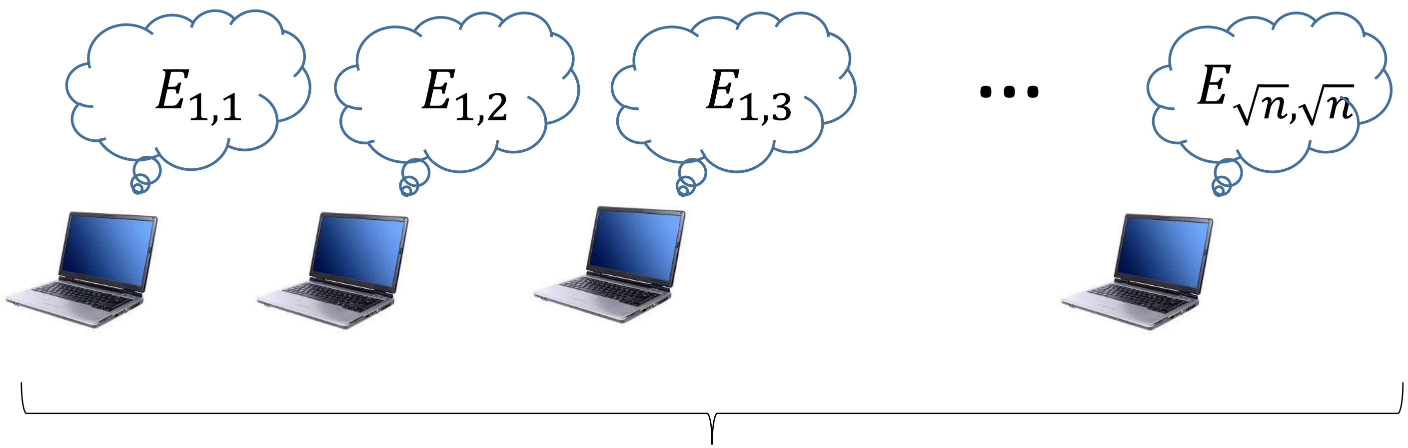
Properties: 1. n different subsets $E_{i,j}$

2. $\left|E_{i,j}\right| = O(n)$

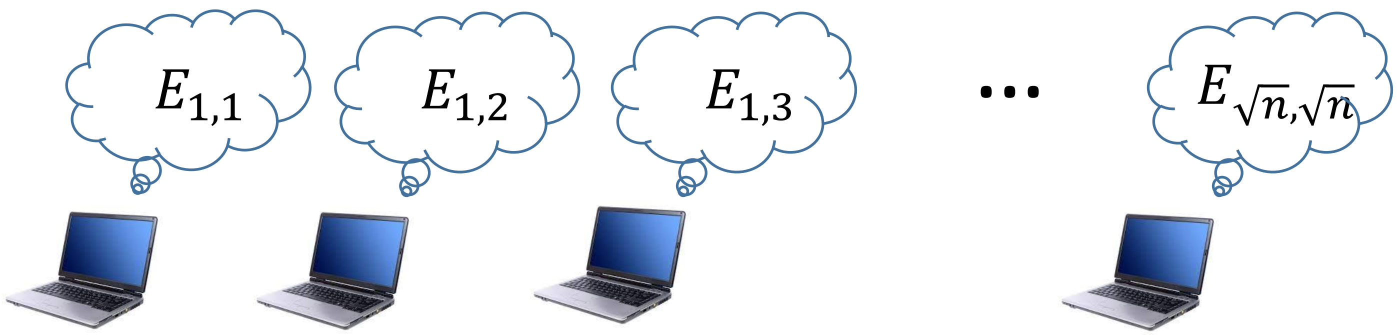
Edge Partitioning: Try 1



n sets of size $O(n)$

Each machine computes a spanner for one set $E_{i,j}$

Edge Partitioning: Try 1



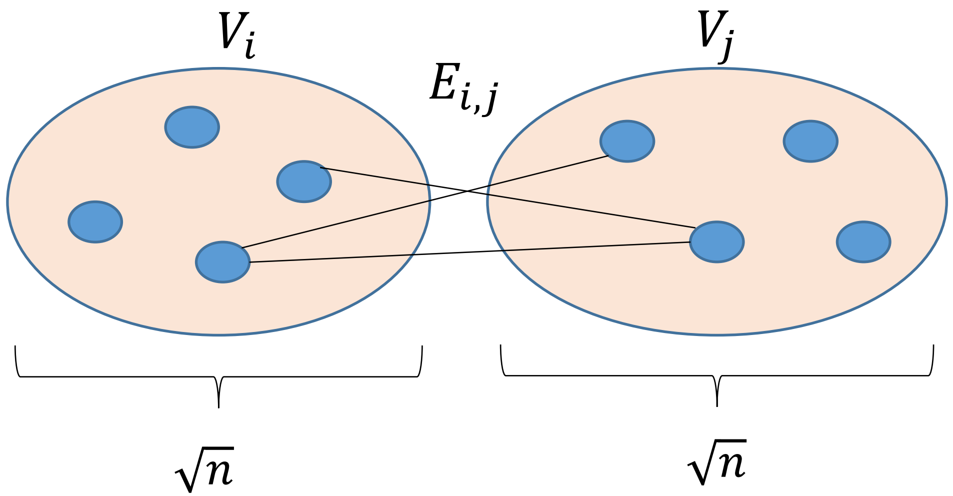
n sets of size $O(n)$

Each machine computes a spanner for one set $E_{i,j}$

What is the size of the spanner?

Edge Partitioning: Try 1

For one set $E_{i,j}$ construct a spanner of size $O(\sqrt{n})$



<u>Overall:</u> $O(n \cdot \sqrt{n}) = O(n^{3/2})$

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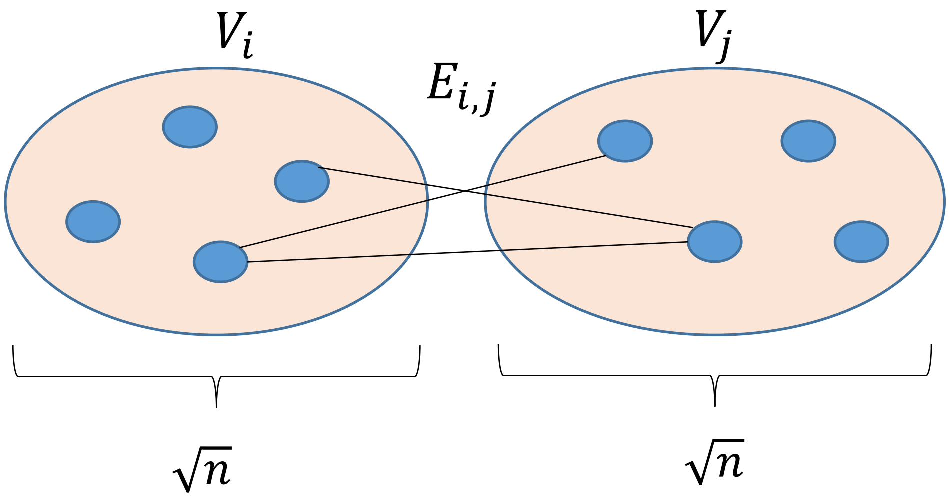

<u>Overall:</u> $O(n \cdot \sqrt{n}) = O(n^{3/2})$ ➡ Too expensive

<u>Goal</u>: $O(n)$ size

Edge Sparsification

<u>Sparsification theorem:</u> Let G be a graph with N vertices and M edges, we can construct $O(\log N)$-spanner for G with $O(N^{2/3}M^{1/3})$ edges

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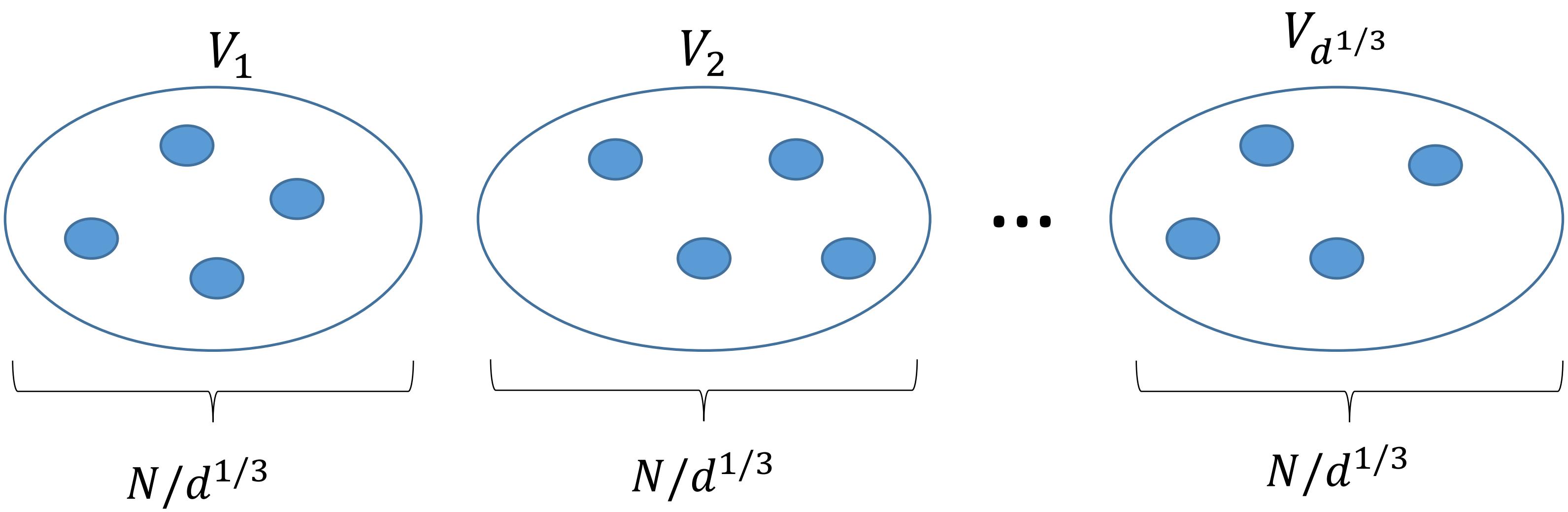
$d = M/N$



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This is still larger than $O(n)$ if $m = \omega(n)$

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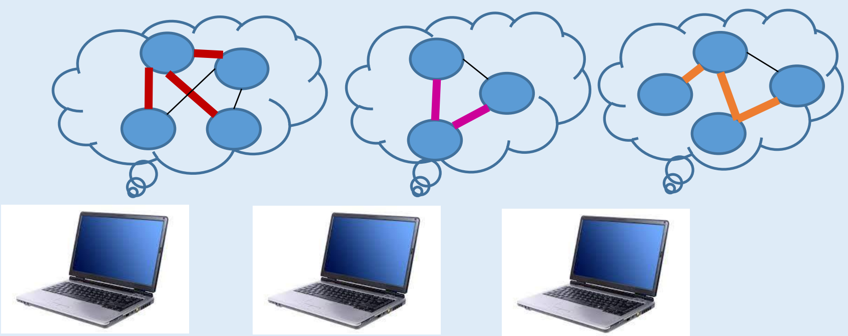
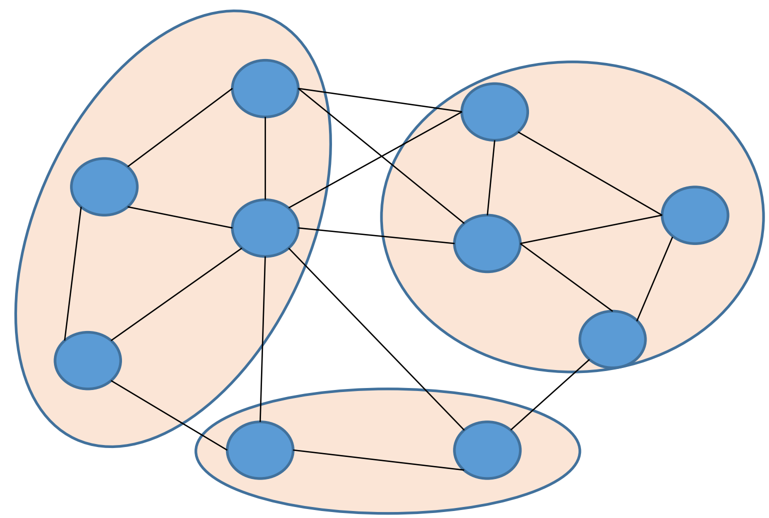
Goal: apply it on a graph where $O(N^{2/3}M^{1/3}) = O(n)$

The Spanner Algorithm

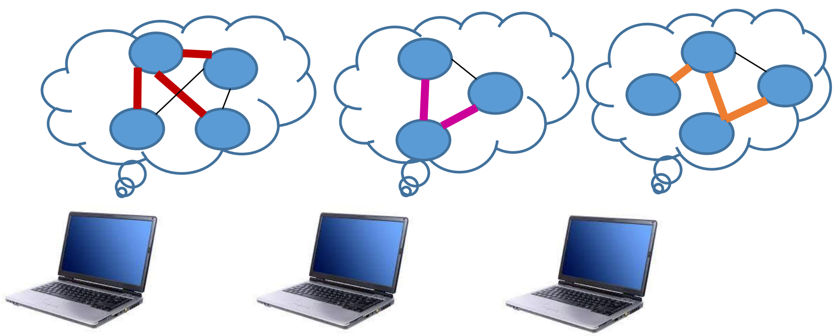
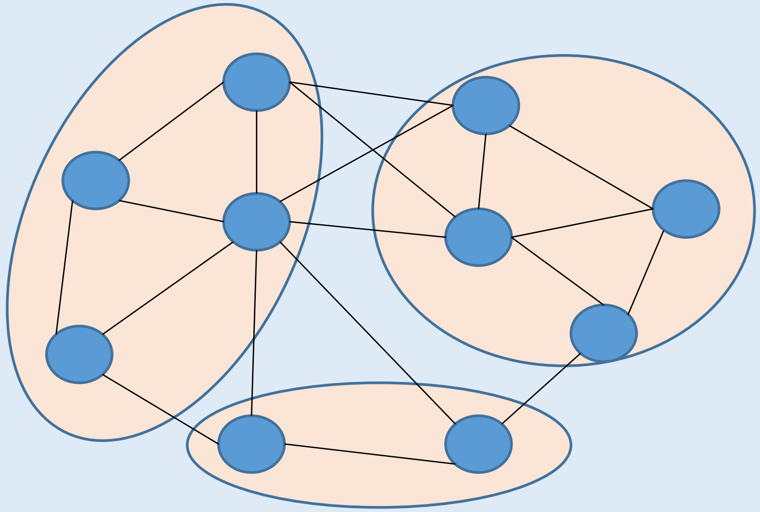
Edge Sparsification



Vertex Sparsification

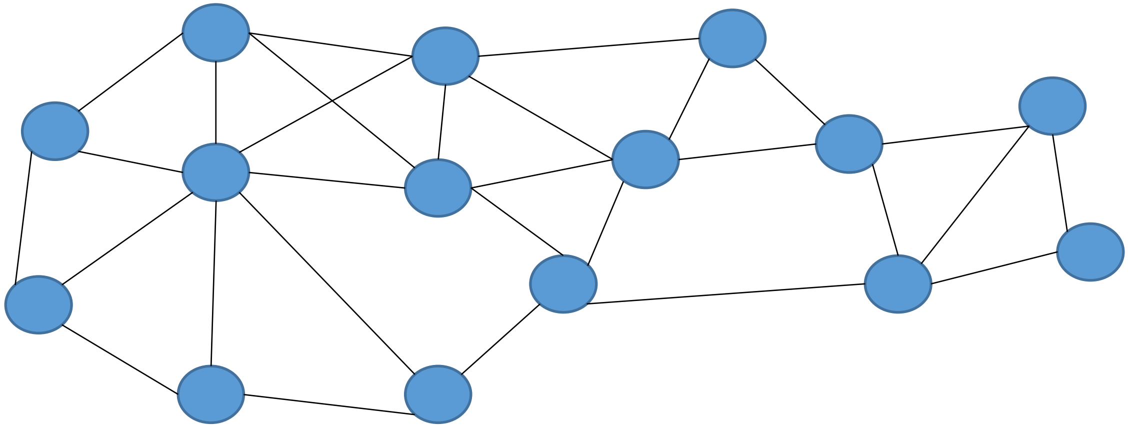


The Spanner Algorithm

Edge Sparsification	Vertex Sparsification
	

Vertex Sparsification

<u>Idea:</u> divide the vertices into groups of close-by vertices, and treat each group as one vertex



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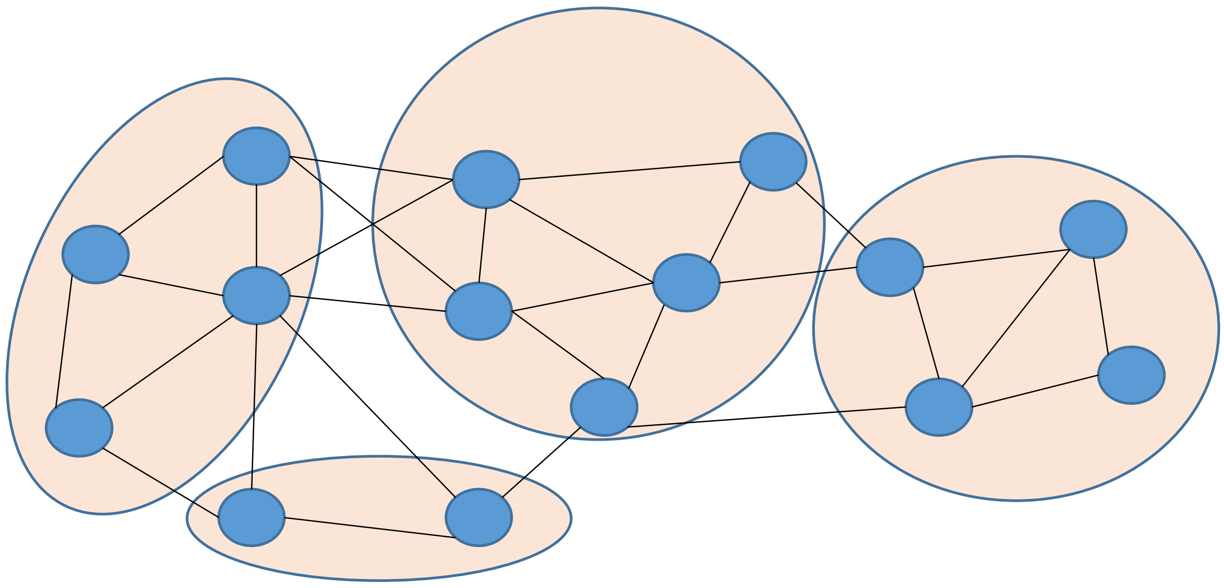

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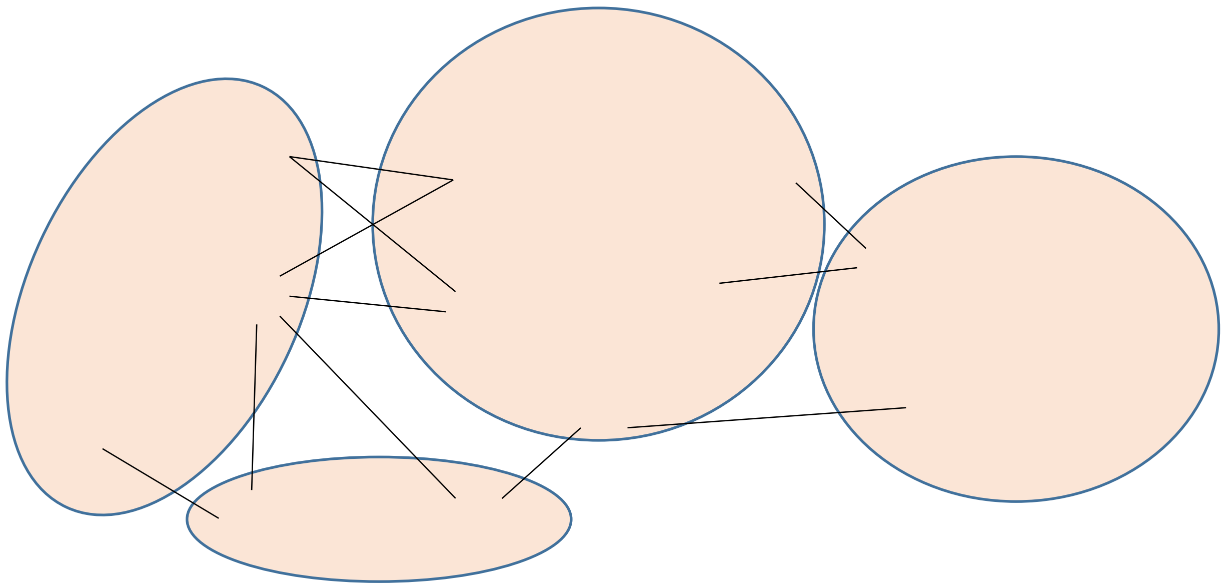
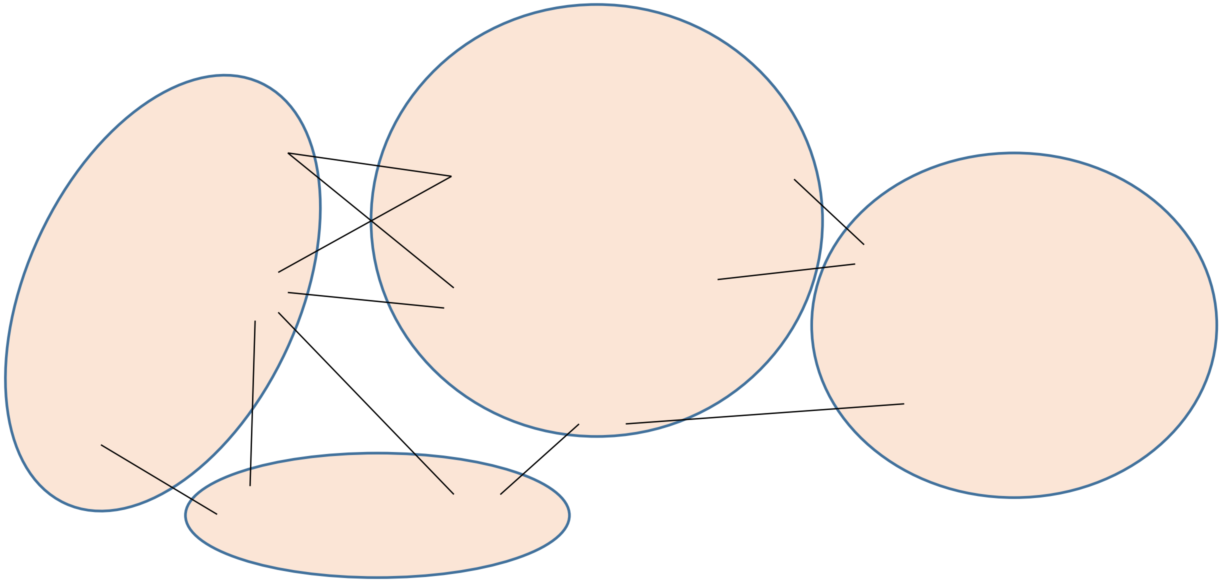


Vertex Sparsification

Using the sparsification theorem: construct $O(n)$ size spanner for the contracted graph



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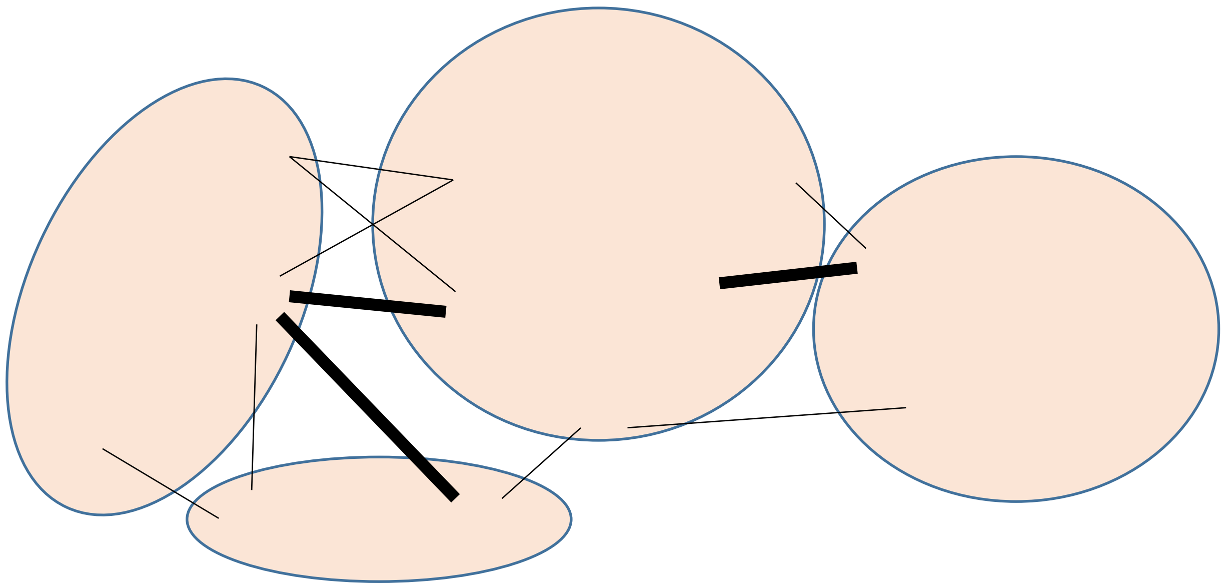
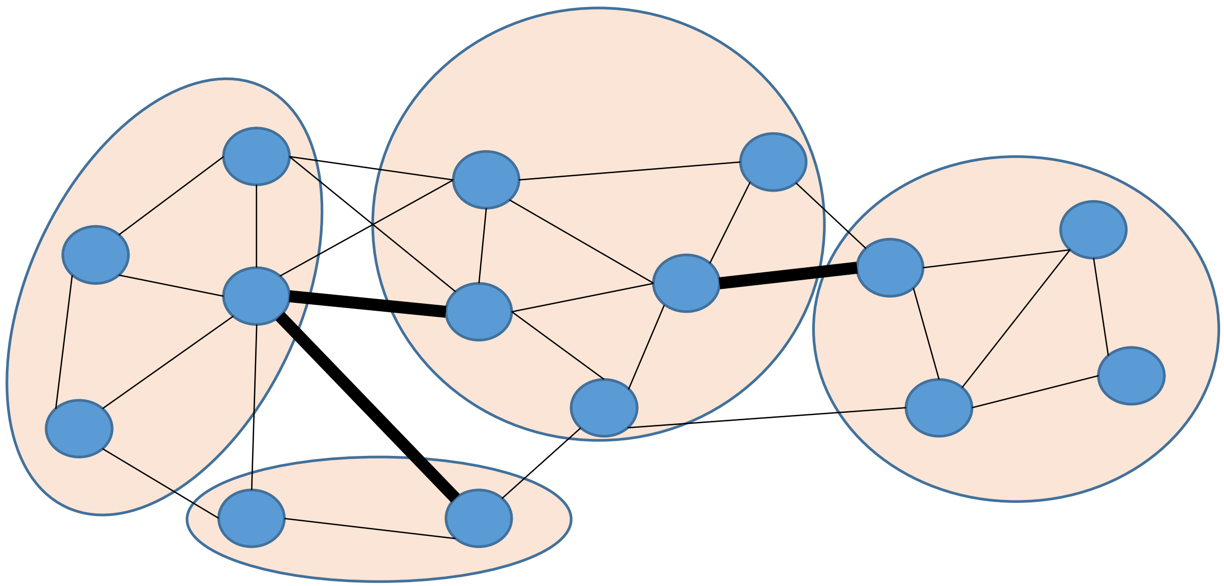


Vertex Sparsification

Can be converted to a spanner for the original graph



Vertex Sparsification

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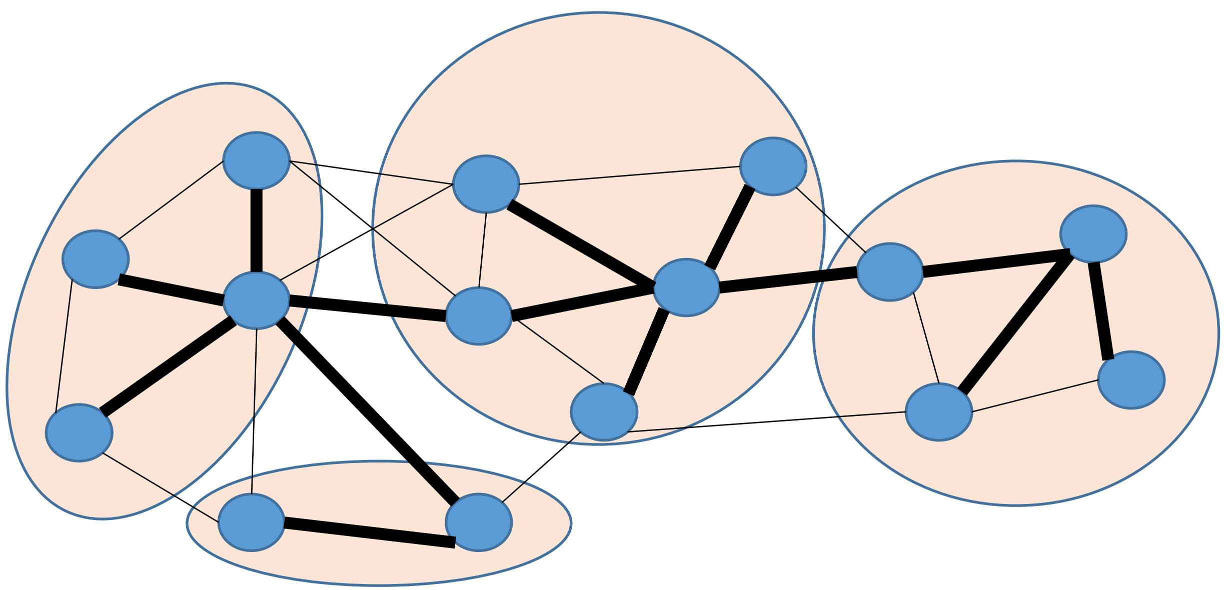
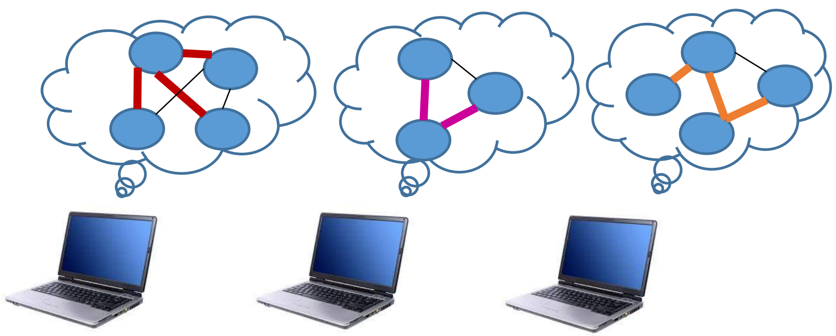
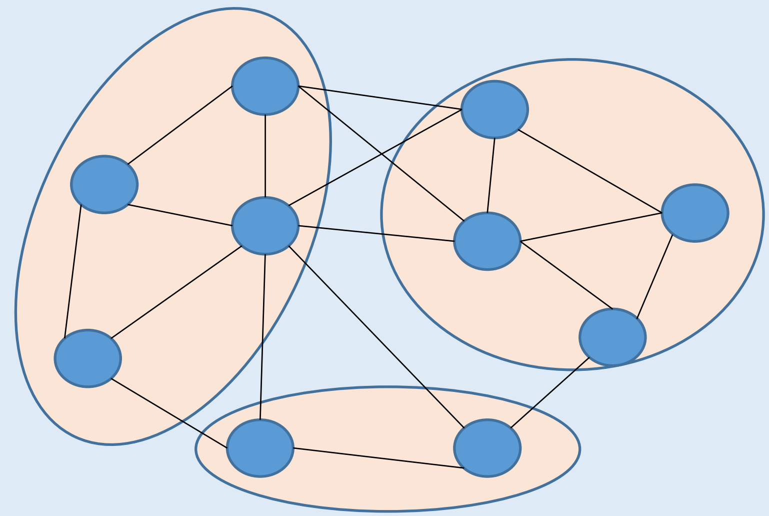


The Spanner Algorithm

Edge Sparsification



Vertex Sparsification



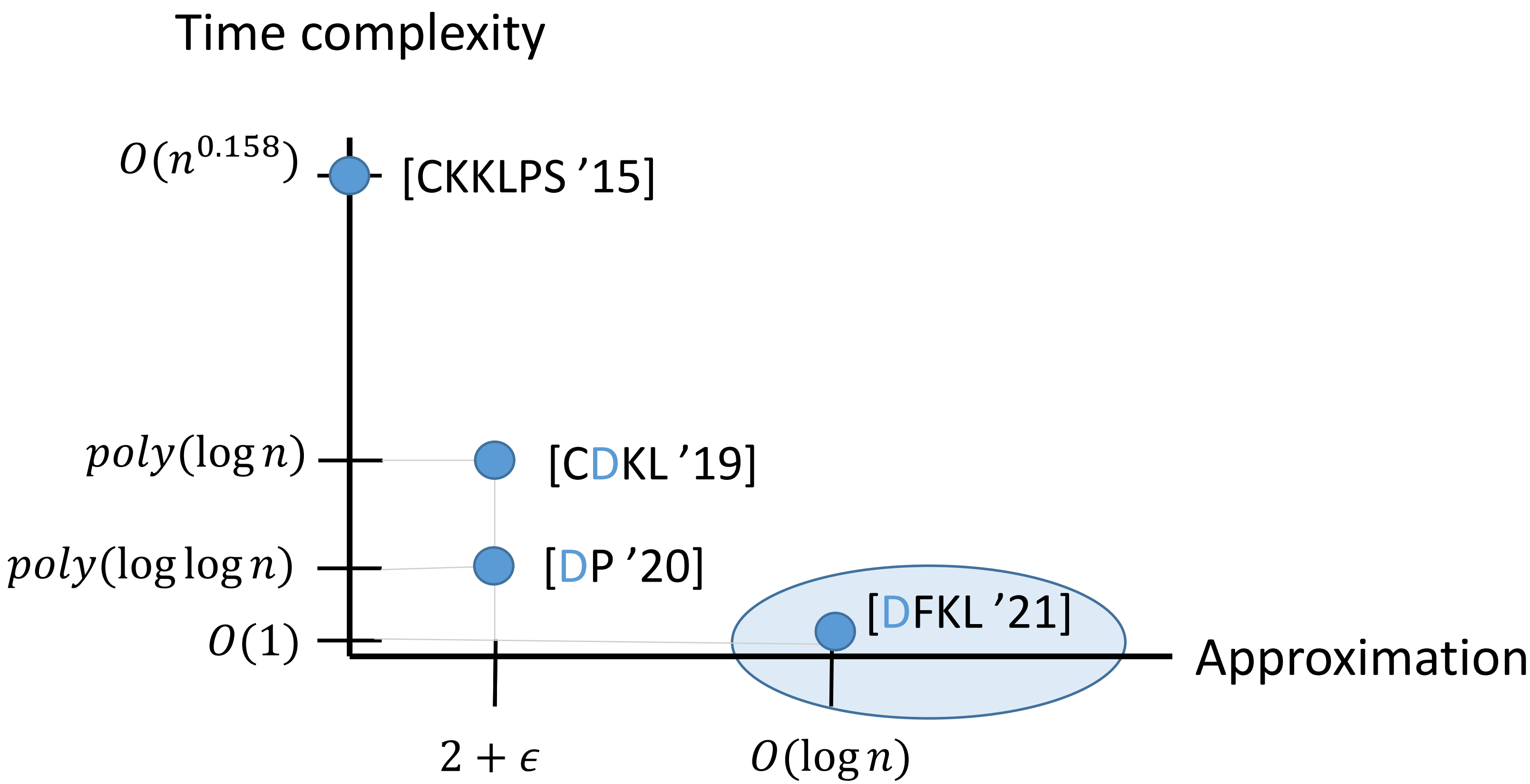
Conclusion

$O(\log n)$-spanners of size $O(n)$

$\Downarrow$

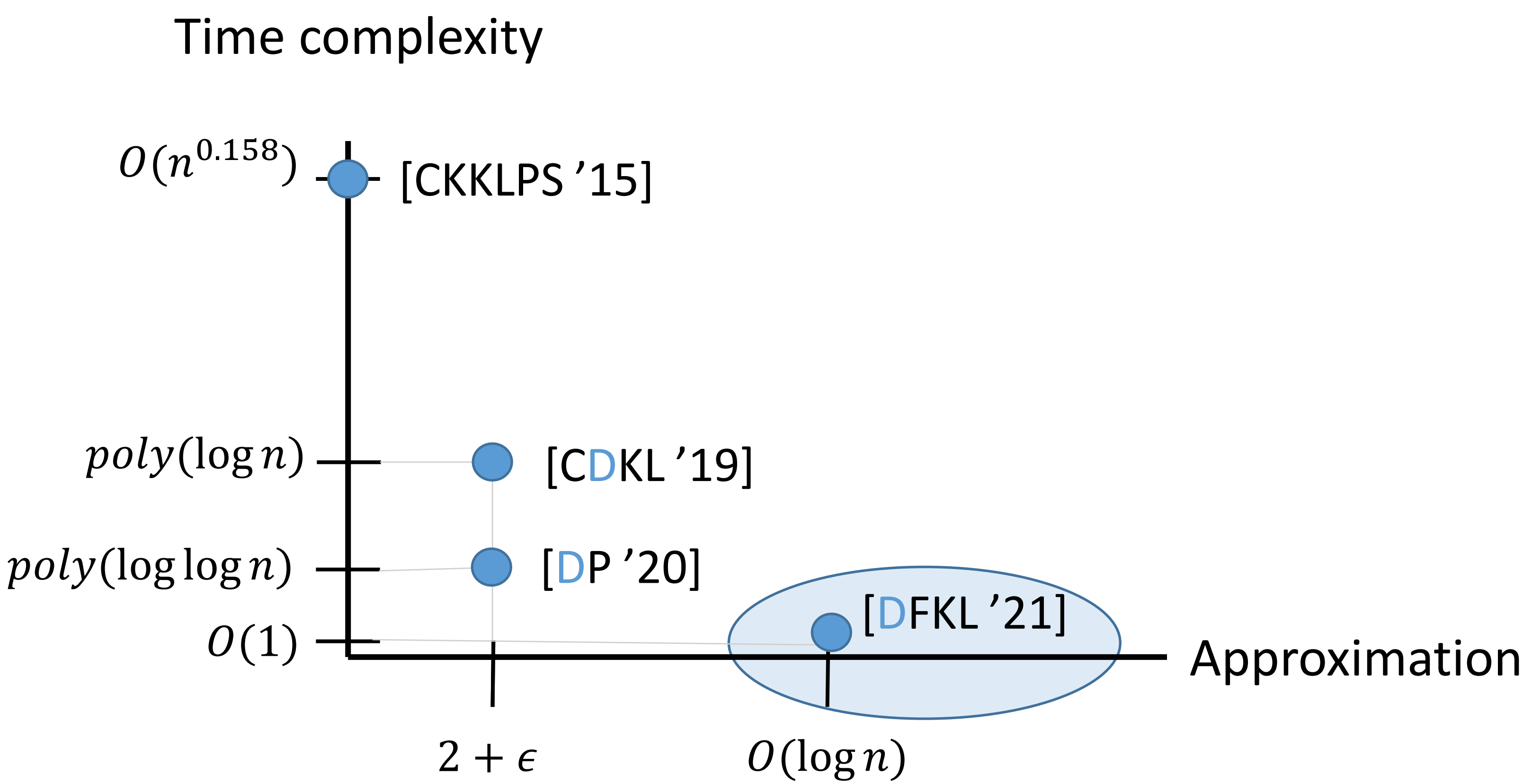
$O(\log n)$-approximation for APSP in $O(1)$ rounds

Distributed APSP



Distributed APSP

How to get a better approximation?



Distributed APSP

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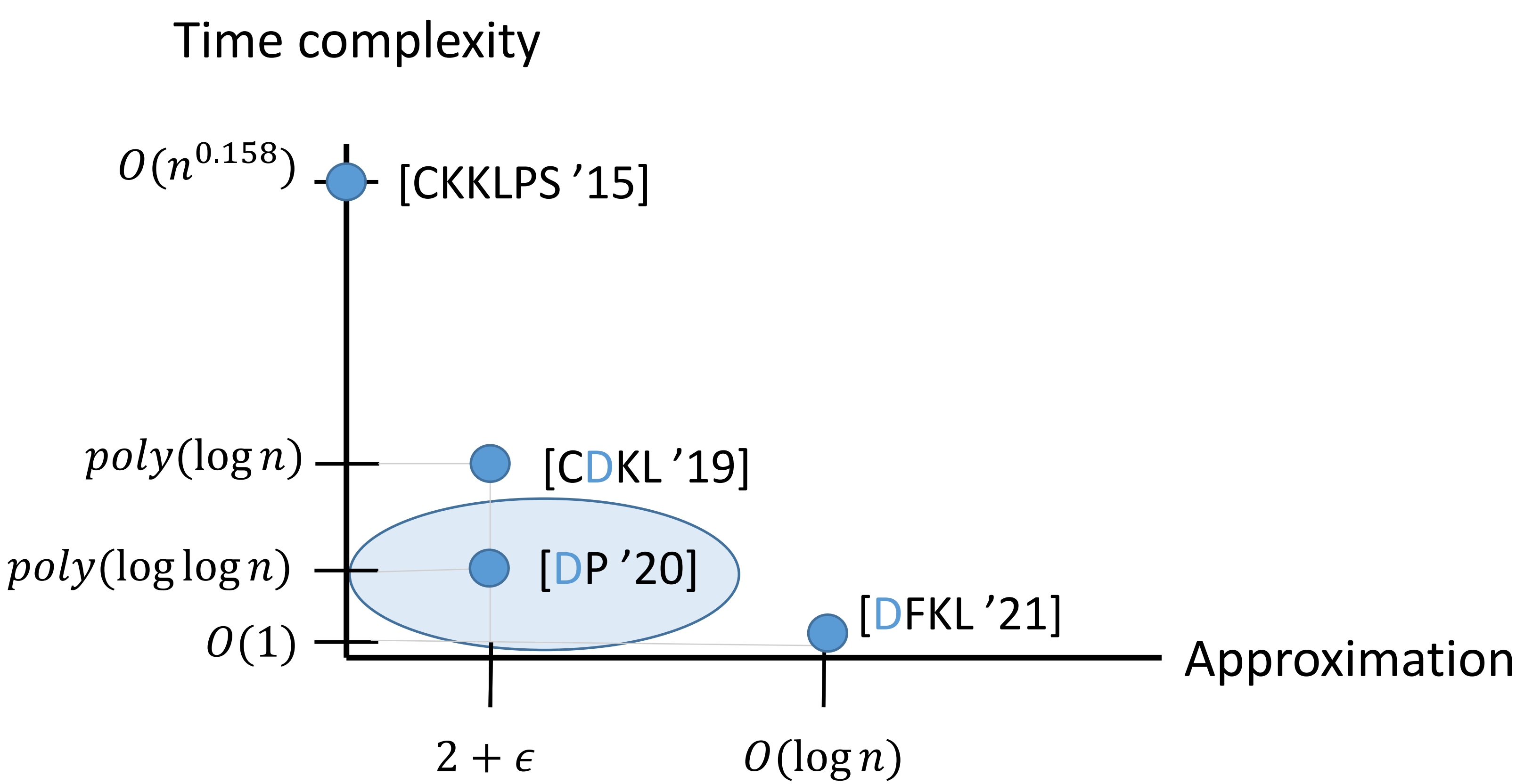
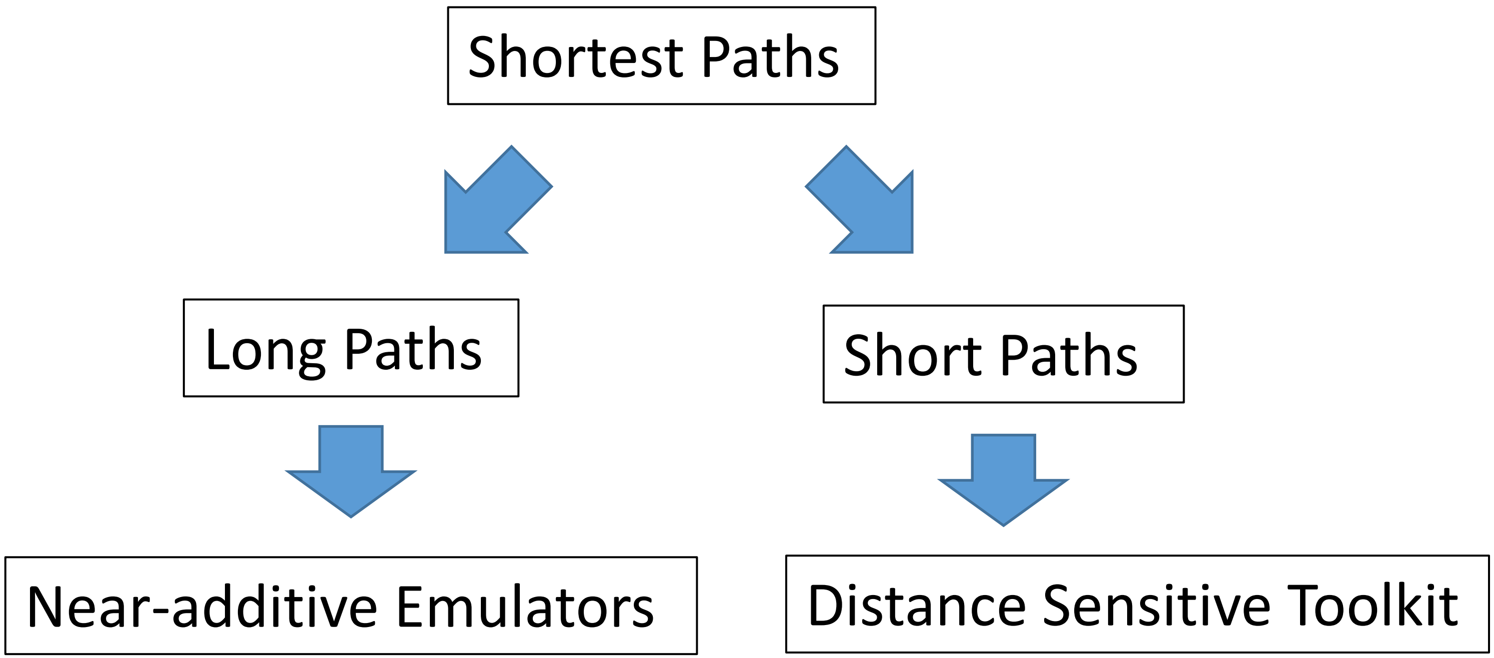


Our Techniques



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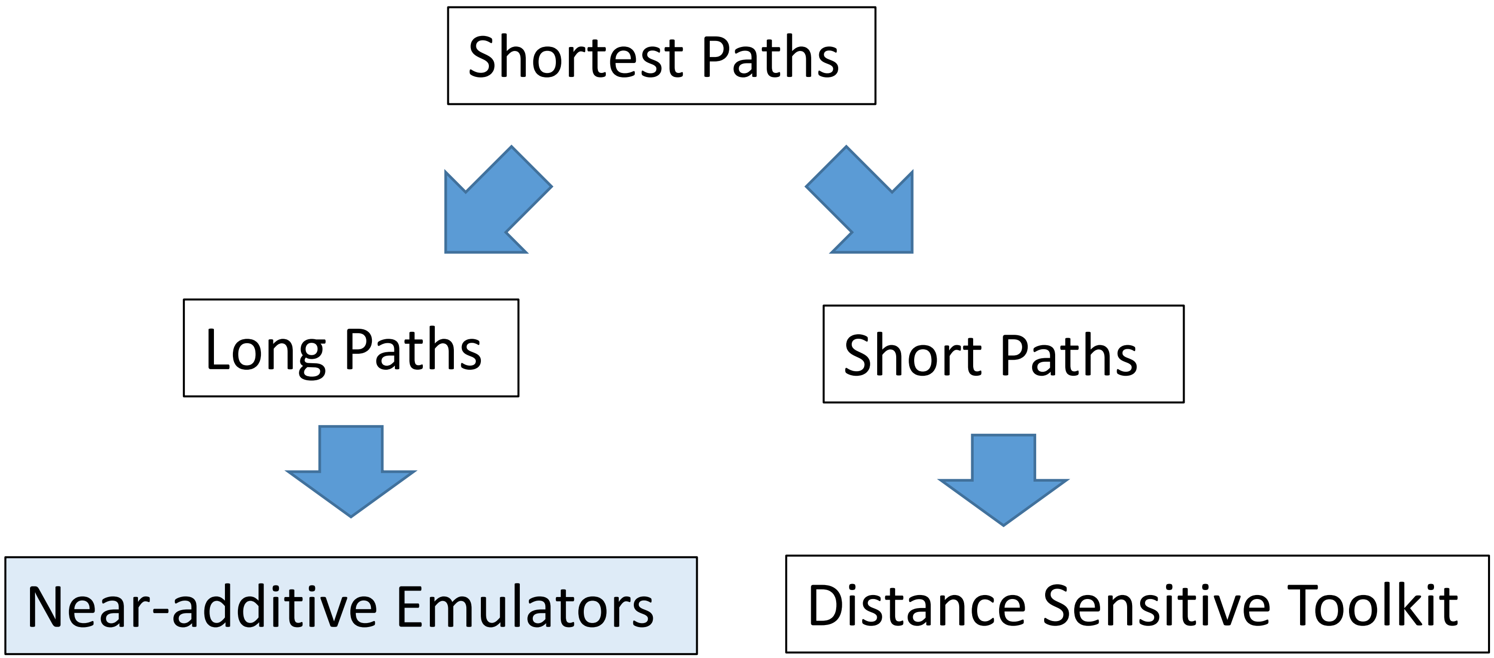

Near-additive Emulator

A sparse graph H such that for all u, v:

$$d(u, v) \leq d_H(u, v) \leq (1 + \epsilon)d(u, v) + \beta$$

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$(1 + \Theta(\epsilon))$-approximation for long distances!

Near-additive Emulator

A sparse graph H such that for all u, v:

$$d(u,v) \leq d_H(u,v) \leq (1+\epsilon)d(u,v) + \beta$$

Build a sparse near-additive emulator

$(1 + \Theta(\epsilon))$-approximation for long distances!

Shortest Paths via Emulators

Left with short paths of length $t = O(\beta/\epsilon)$

$\boxed{\text{Requires } poly(\log t) = poly(\log\log n) \text{ time!}}$

Shortest Paths via Emulators

Shortest Paths

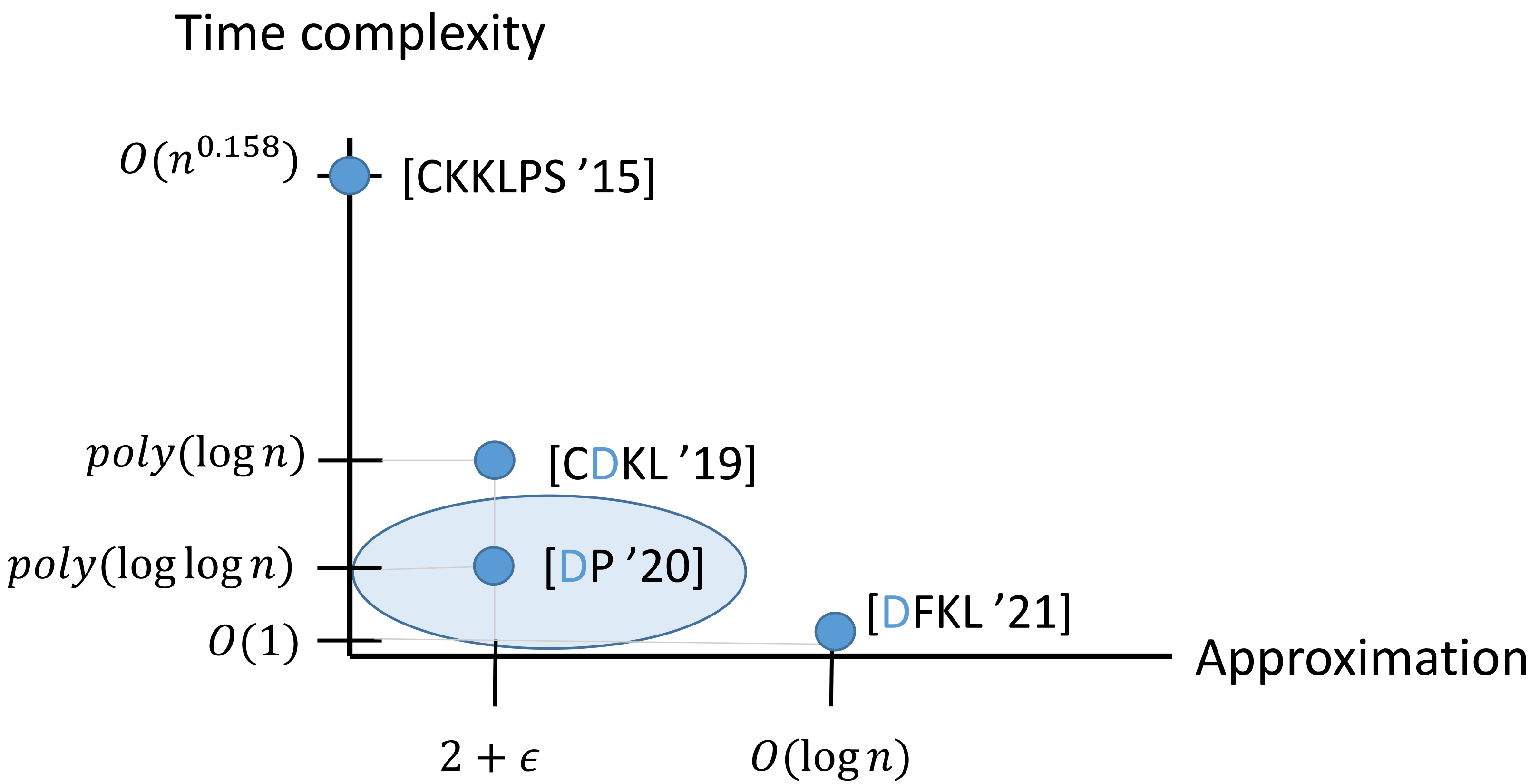
Long Paths: $\Omega(\beta/\epsilon)$

Short Paths: $O(\beta/\epsilon)$

Near-additive Emulators:
$(1+\Theta(\epsilon))$-approximation

Distance Sensitive Toolkit:
$poly\left(\log\frac{\beta}{\epsilon}\right)$ time

Distributed APSP



Conclusion

We saw:

Round Complexity	**Approximation**
$poly(\log\log n)$	$2+\epsilon$
$O(1)$	$O(\log n)$

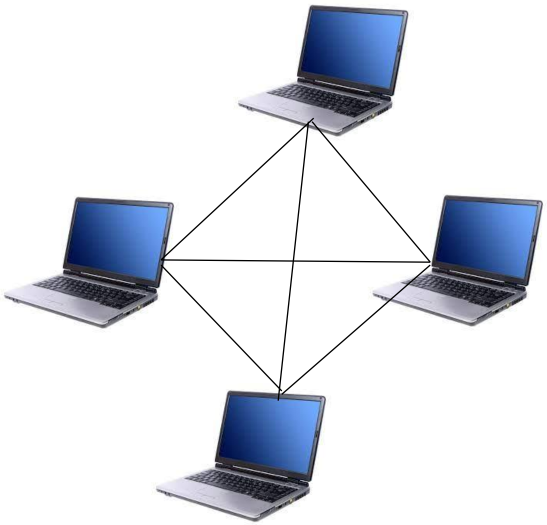
Conclusion

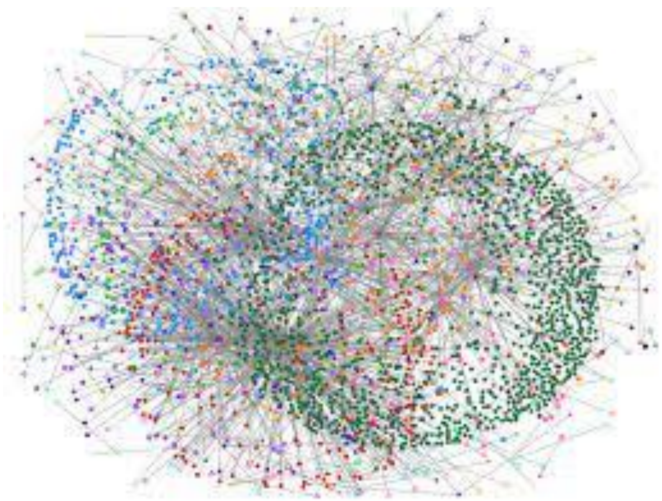
We saw:

Round Complexity	Approximation
$poly(\log \log n)$	$2 + \epsilon$
$O(1)$	$O(\log n)$

Can we get $O(1)$-approximation in $O(1)$ rounds?

The Model - Congested Clique

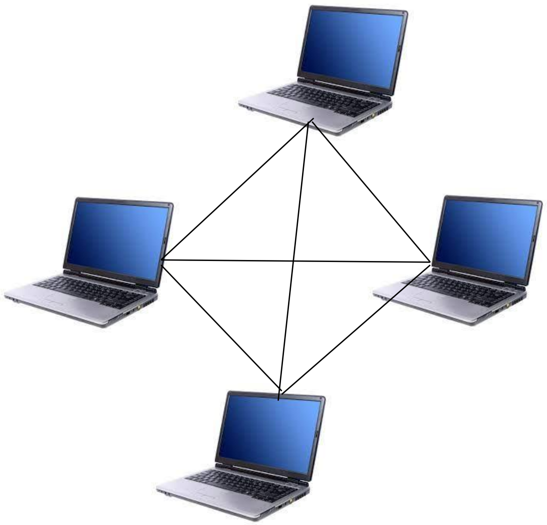




n vertices

- n machines
- Input per machine: $O(n)$ edges
- Each machine can send/receive a total of n messages of $O(\log n)$ bits per round

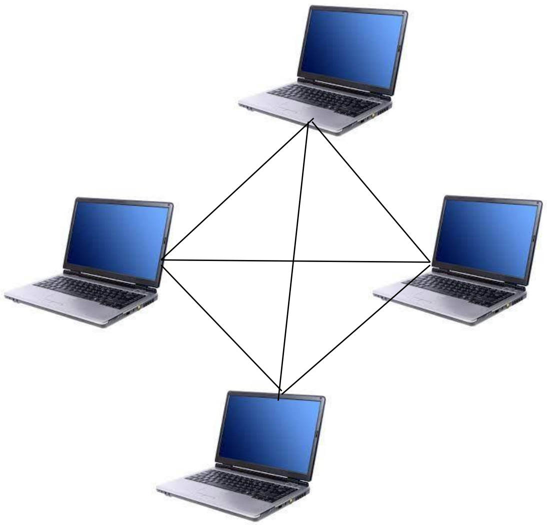
The Model - Congested Clique



What happens if we bound the memory?

- n machines
- Input per machine: $O(n)$ edges
- Each machine can send/receive a total of n messages of $O(\log n)$ bits per round

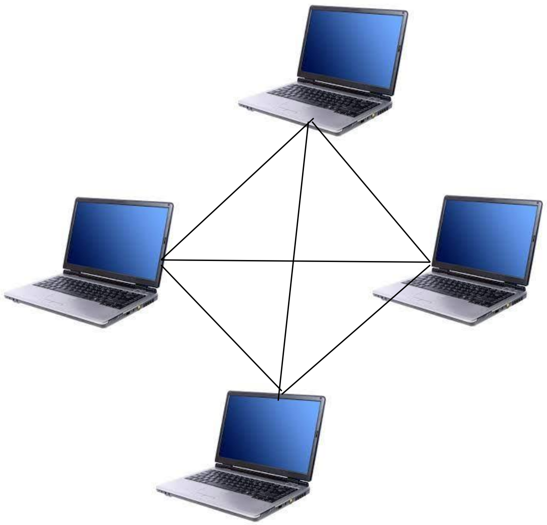
The Model – Linear Memory MPC



What happens if we bound the memory?

- $O(m/n)$ machines – total memory of $\tilde{O}(m)$
- $\tilde{O}(n)$ memory per machine

The Model – Linear Memory MPC

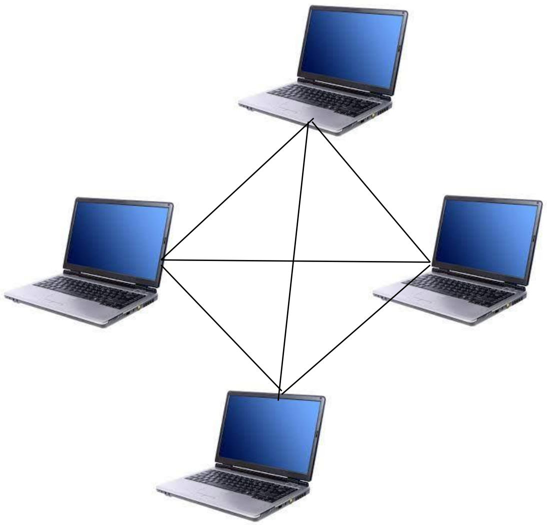


☑ $O(\log n)$-approximation

☒ $O(1)$-approximation

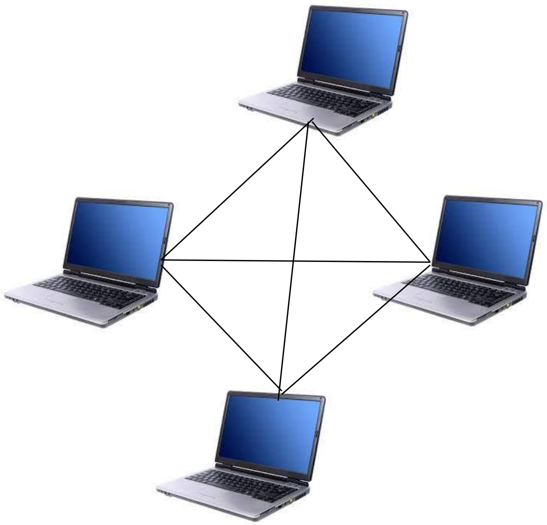
- $O(m/n)$ machines – total memory of $\tilde{O}(m)$
- $\tilde{O}(n)$ memory per machine

The Model – Sublinear Memory MPC



- $O(m/n^{\gamma})$ machines – total memory of $\tilde{O}(m)$
- $\tilde{O}(n^{\gamma})$ memory per machine, $\gamma < 1$

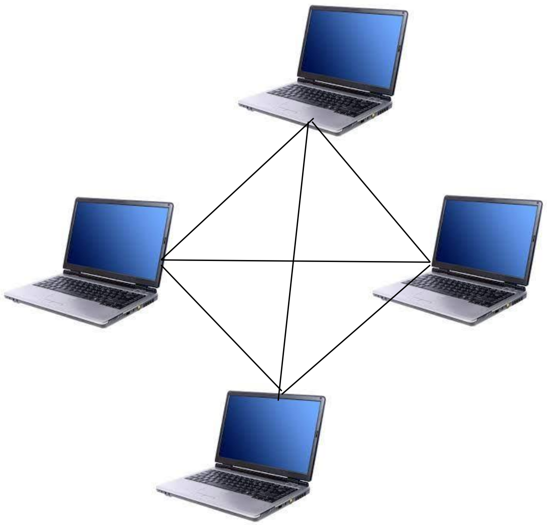
The Model – Sublinear Memory MPC



- Spanners in $poly(\log\log n)$ time [BDGMN, 2021]
- Conditional $\Omega(\log n)$ lower bounds for shortest paths computations

- $O(m/n^{\gamma})$ machines – total memory of $\tilde{O}(m)$
- $\tilde{O}(n^{\gamma})$ memory per machine, $\gamma < 1$

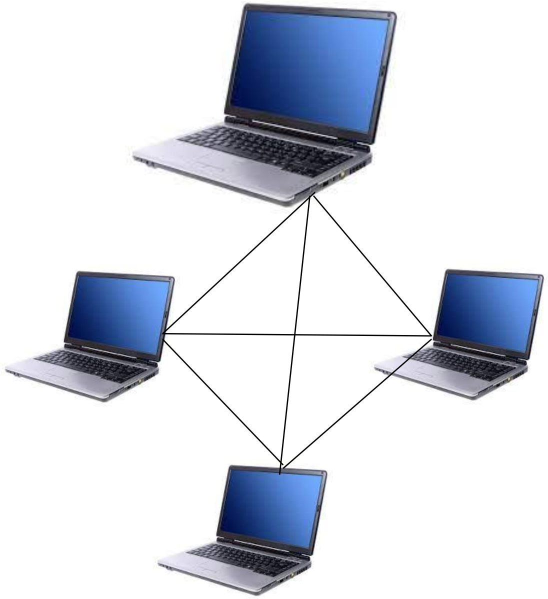
The Model – Sublinear Memory MPC



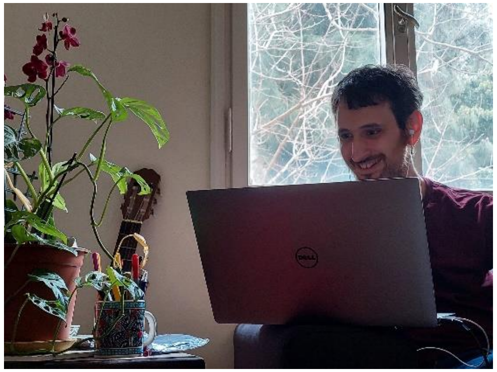
<u>See Friday's AMG talk</u>:
Yasamin Nazari – Distance Computation in MPC and related models

- $O(m/n^{\gamma})$ machines – total memory of $\tilde{O}(m)$
- $\tilde{O}(n^{\gamma})$ memory per machine, $\gamma < 1$

The Model –Heterogeneous MPC



Next ADGA talk:
Orr Fischer - Massively Parallel Computation in a Heterogeneous Regime: One Strong Machine Makes a Big Difference



One linear machine,
The rest – sublinear machines

Open Questions

- Can we get $O(1)$-approximation in $O(1)$ rounds?
- Directed/Exact shortest paths
- What other problems can be solved in $O(1)$ rounds?

Open Questions

<u>$O(1)$-round algorithms</u>:
- Approximate APSP
- Spanners
- Minimum Spanning Tree
- $(\Delta + 1)$-Coloring

<u>$O(\log \log n)$-round algorithms</u>:
- Approximate Maximum Matching
- Maximal Independent Set
- Approximate Vertex Cover