Speedup Theorems for All

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Speedup Theorem

Hypothetic Generic Speedup Theorem

Theorem Let $\mathscr{M}$ be a distributed computing model. There exists a function

$$F : \{\text{problems}\} \rightarrow \{\text{problems}\}$$

such that, for every $t \geq 0$, and every problem Π,

$$\Pi \text{ has complexity } t \iff F(\Pi) \text{ has complexity } t - 1.$$

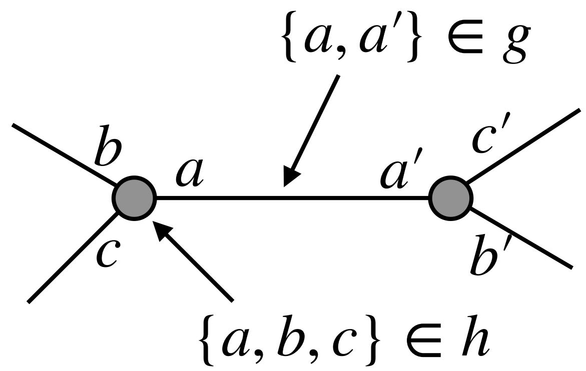
Corollary Π has complexity $t \iff F^{(t)}(\Pi)$ has complexity 0.

Brandt's Speedup Theorem

[PODC 2019]

- $\mathscr{M}$ = anonymous LOCAL model in $\mathscr{G}_\Delta = \{G : \deg(G) \leq \Delta\}$

- Locally Checkable Labeling (LCL): (f, g, h)
  - f is a finite set of labels
  - g is a collection of pairs of labels
  - h is a collection of multisets of labels



- Problem Π:
  - Input: labeling in (f_{in}, g_{in}, h_{in})
  - Output: labeling in $(f_{out}, g_{out}, h_{out})$

$$F(f, g, h) = (f', g', h')$$

- $f' = 2^{2^f}$ is a set of sets of labels
- $\{\mathbf{S}, \mathbf{S}'\} \in g'$ if and only if

$$\exists (S, S') \in \mathbf{S} \times \mathbf{S}', \forall (s, s') \in S \times S' : \{s, s'\} \in g$$

- $\{\mathbf{S}_1, \ldots, \mathbf{S}_\Delta\} \in h'$ if and only if

$$\forall (S_1, \ldots, S_\Delta) \in \mathbf{S}_1 \times \ldots \times \mathbf{S}_\Delta,$$
$$\exists (s_1, \ldots, s_\Delta) \in S_1 \times \ldots \times S_\Delta : \{s_1, \ldots, s_\Delta\} \in h$$

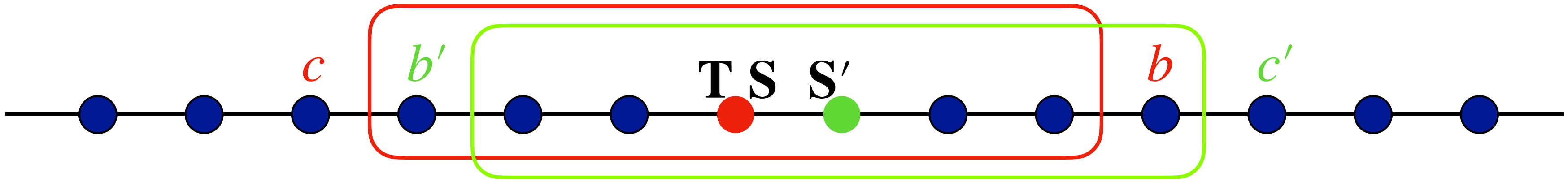
Theorem [Brandt, 2019] For every $t \geq 1$, and for every LCL problem $\Pi = \big((f_{in}, g_{in}, h_{in}), (f_{out}, g_{out}, h_{out})\big)$,

$(f_{out}, g_{out}, h_{out})$ constructible in t rounds from (f_{in}, g_{in}, h_{in})

$$\Updownarrow$$

$F(f_{out}, g_{out}, h_{out})$ constructible in $t-1$ rounds from (f_{in}, g_{in}, h_{in}).

Proof (for $\Delta = 2$)



$F(\Pi)$ in $t-1$ rounds $\Rightarrow$ Π in t rounds:

- $\{\mathbf{S}, \mathbf{S}'\} \in g'_{out} \iff \exists (S, S') \in \mathbf{S} \times \mathbf{S}', \forall (s, s') \in S \times S', \{s, s'\} \in g_{out}$
- $\{\mathbf{S}, \mathbf{T}\} \in h'_{out} \iff \forall (S, T) \in \mathbf{S} \times \mathbf{T}, \exists (s, t) \in S \times T, \{s, t\} \in h_{out}$

Π in t rounds $\Rightarrow$ $F(\Pi)$ in $t-1$ rounds:

- $(t-1)$-round view $w = (a_{-t+1}, \ldots, a_{-1}, a_0, a_1, \ldots, a_{t-1}) \in f_{in}^{2t-1}$
- For every $b \in f_{in}$, let $S_b = \{\mathrm{out}(c, w, b) : c \in f_{in}\}$
- Set $\mathbf{S} = \{S_b : b \in f_{in}\}$

Lower Bounds

- Nathan Linial [FOCS 1987]:

  Lower bound $\frac{1}{2}\log^{\star} n$ rounds for 3-coloring C_n

- Sebastian Brandt [PODC 2019]:

  Formalization of speedup theorem in anonymous LOCAL model

- Alkida Balliu, Sebastian Brandt, Juho Hirvonen, Dennis Olivetti, Mikaël Rabie, Jukka Suomela [FOCS 2019]:

  Lower Bounds for Maximal Matchings and Maximal Independent Set.

- Which models admit Speedup Theorems?
    - Full-Information protocols
    - Round-Based
- Which problems admit Speedup Theorems?
    - Definition of tasks

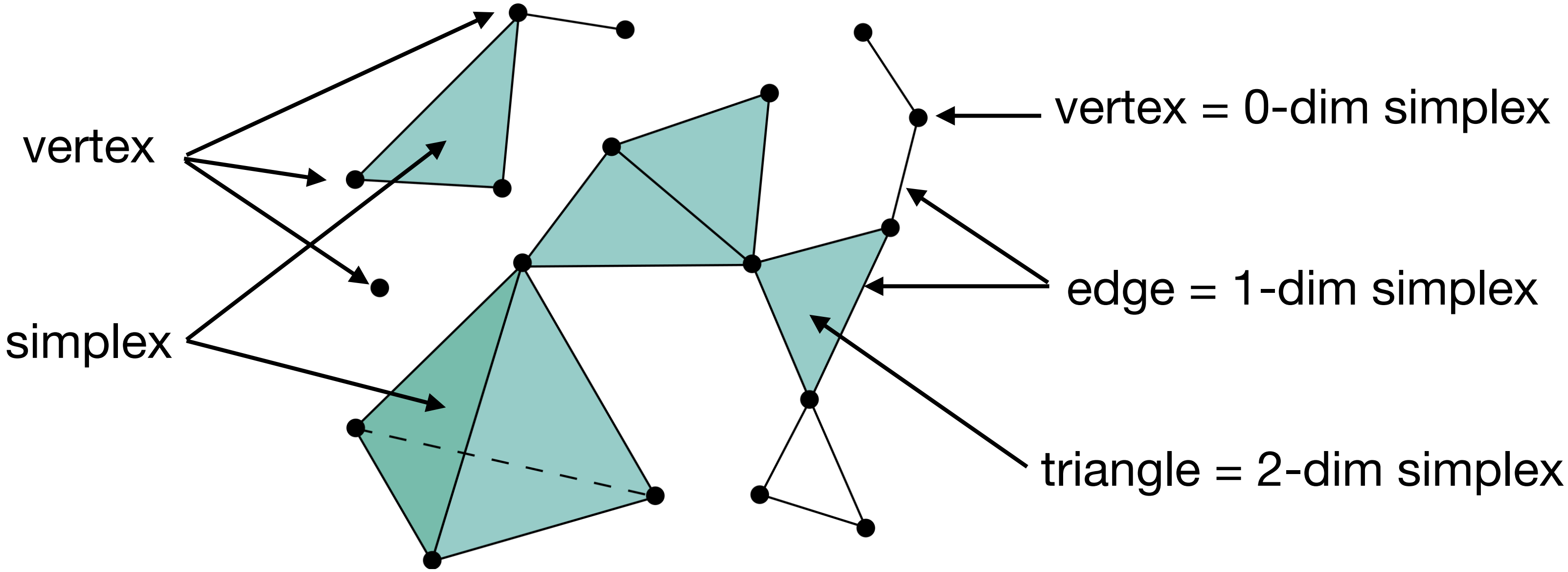
Distributed Computing Through the Lens of Algebraic Topology

Simplicial Complexes

A simplicial complex $\mathscr{K}$ is a pair (V, S) where V is a finite set, $\{\{v\} : v \in V\} \subseteq S \subseteq 2^V \setminus \{\varnothing\}$, and

$$\forall \sigma \in S, \forall \sigma' \subseteq \sigma : \sigma' \neq \varnothing \Rightarrow \sigma' \in S$$

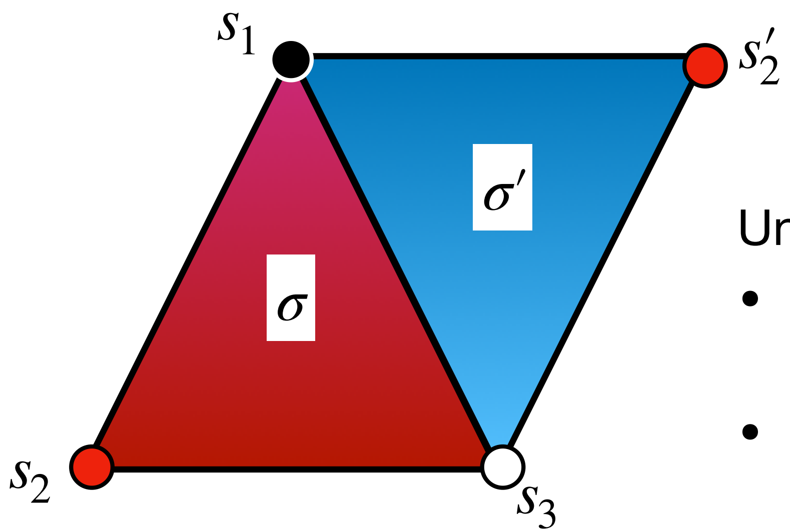
The elements of V are called vertices, and the elements of S are called simplices.



Example taken from wikipedia

Global System States

- Assume n processes, labeled from 1 to n
- Global state $\sigma = \{(i, s_i) : i \in [n]\}$



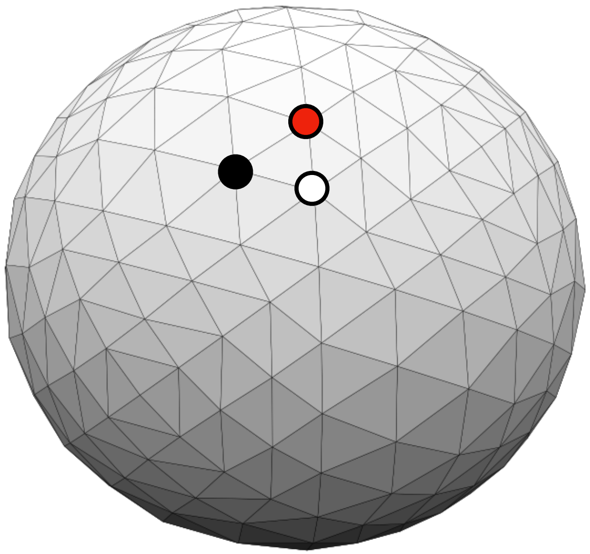
Uncertainty:

- Process ● in state s_1 cannot distinguish σ from σ'
- Even Processes ● and ○ together, in respective states s_1 and s_3, cannot distinguish σ from σ'

Global System States

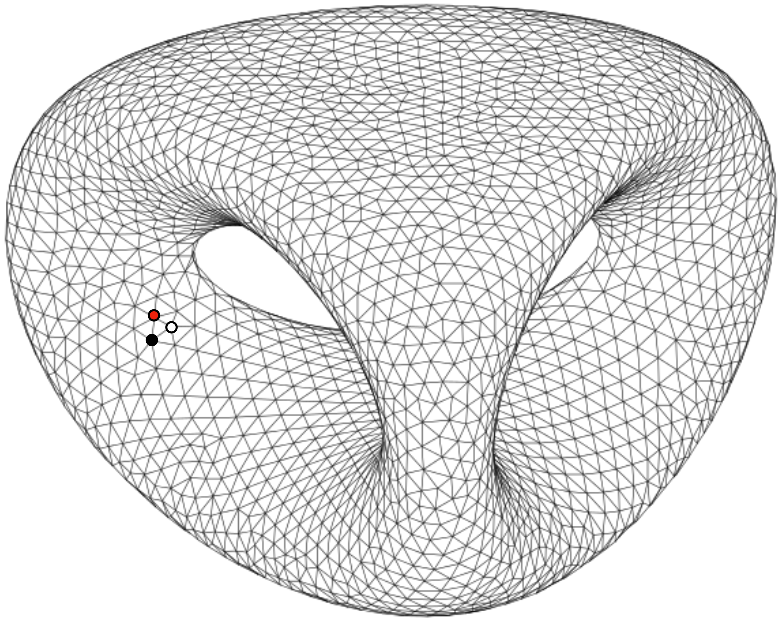
$(n = 3)$

Initial states



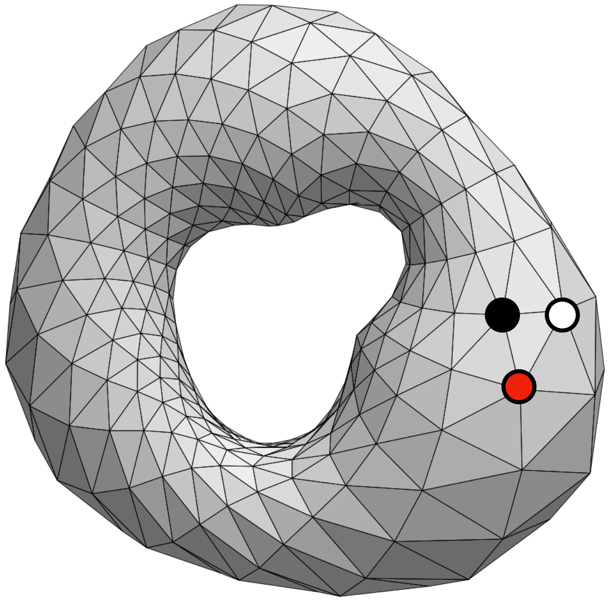
Input Complex $\mathscr{I}$

States at some time $t \geq 1$



Protocol Complex $\mathscr{P}^{(t)}$

Final states

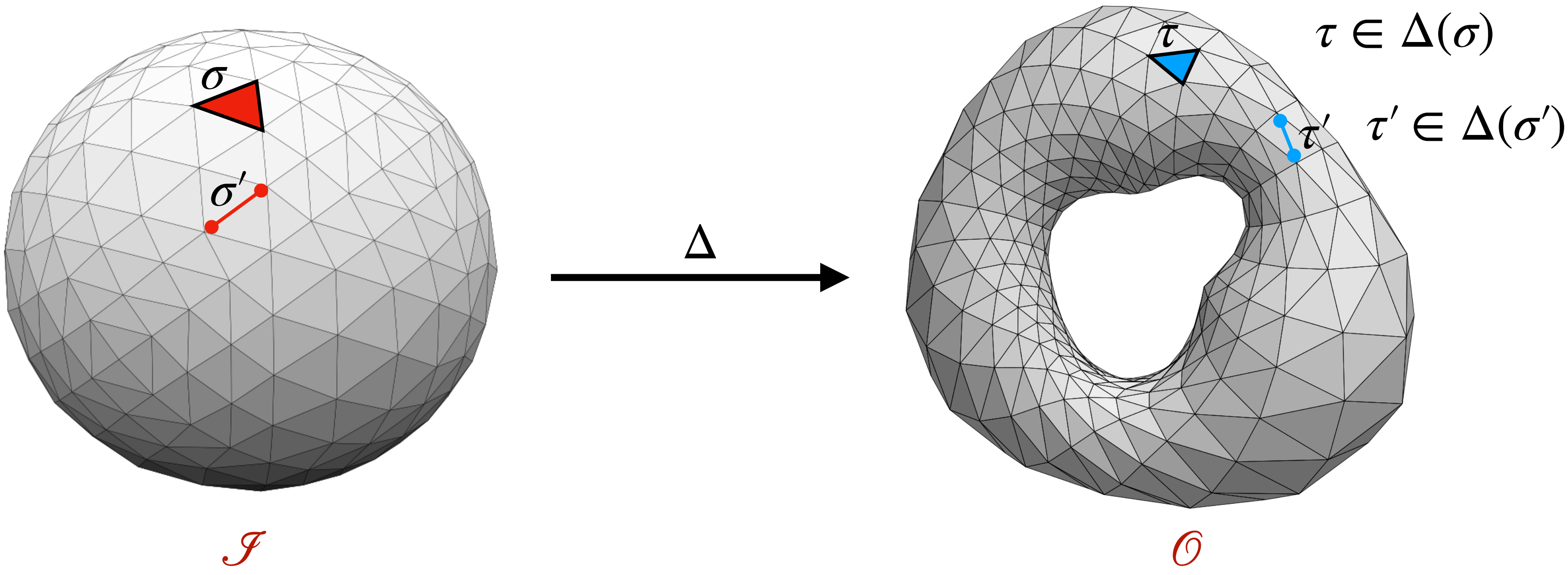


Output Complex $\mathscr{O}$

All these complexes are chromatic

Triangulation taken from wikipedia

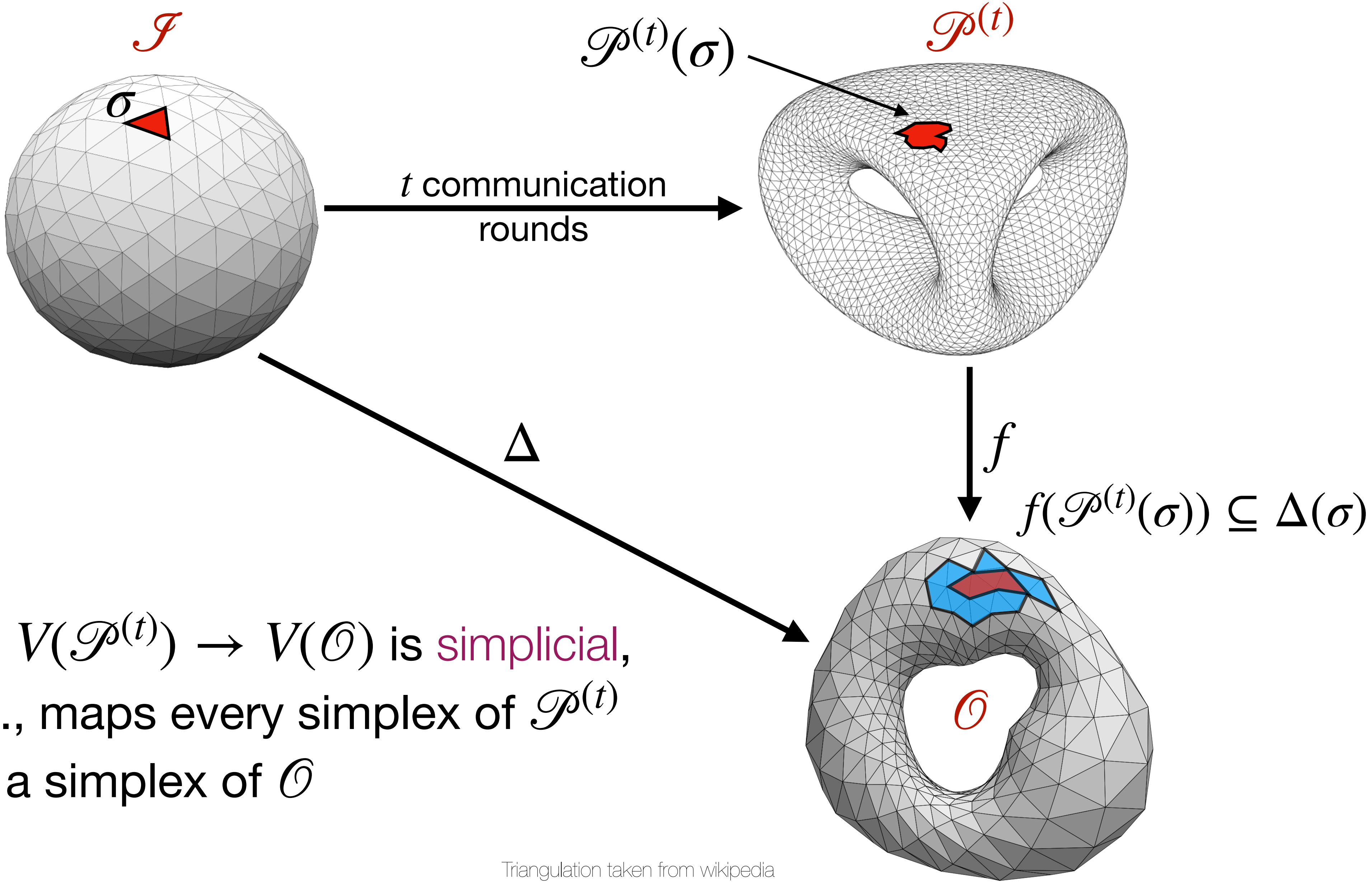
Input-Output Specification



A task is a triple $\Pi = (\mathscr{I}, \mathscr{O}, \Delta)$

Triangulation taken from wikipedia

Solving a Task $\Pi = (\mathscr{I}, \mathscr{O}, \Delta)$



$f : V(\mathscr{P}^{(t)}) \to V(\mathscr{O})$ is simplicial,
i.e., maps every simplex of $\mathscr{P}^{(t)}$
to a simplex of $\mathscr{O}$

Triangulation taken from wikipedia

Task Solvability

Theorem A task $\Pi = (\mathcal{I}, \mathcal{O}, \Delta)$ is solvable in t rounds in Model $\mathcal{M}$ if and only if there exists a chromatic simplicial map

$$f : V(\mathcal{P}^{(t)}) \to V(\mathcal{O})$$

that agrees with Δ, i.e., for every $\sigma \in \mathcal{I}$,

$$f(\mathcal{P}^{(t)}(\sigma)) \subseteq \Delta(\sigma).$$

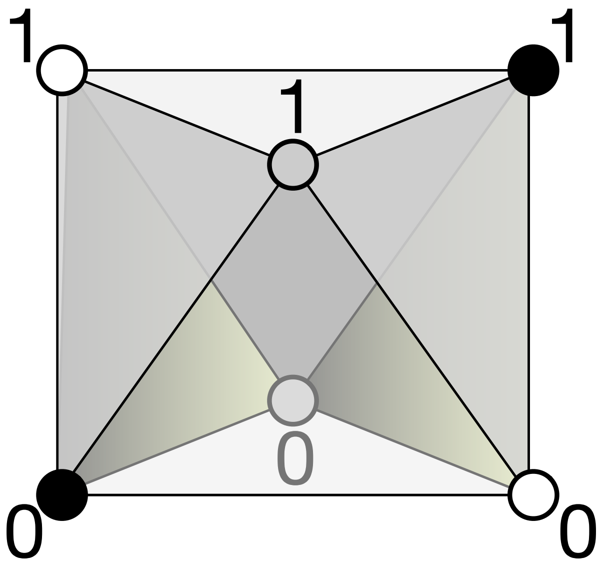
<u>Challenge:</u> Understanding the topological deformation $\mathcal{P}^{(t)}$ of $\mathcal{I}$ after t rounds.

<u>Remark:</u> $\mathcal{P}^{(t)}$ depends on the computing model $\mathcal{M}$.

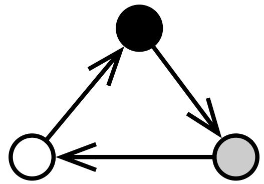
Protocol Complex

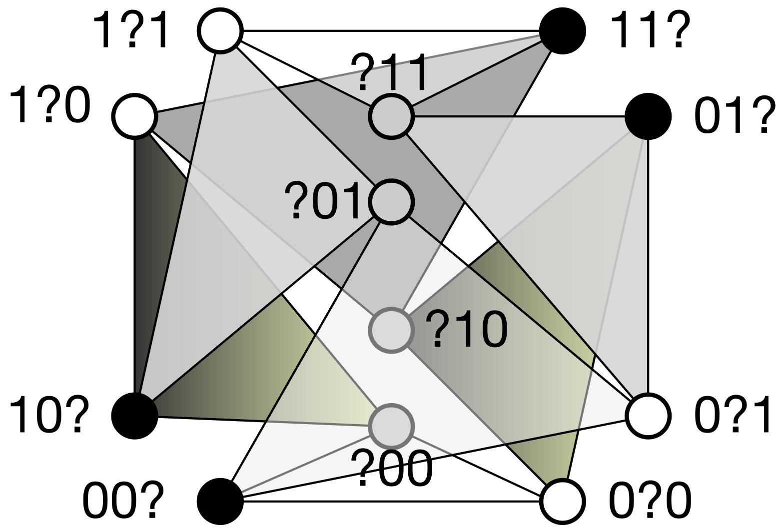
Example 1

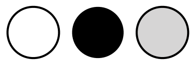
Input
complex



Protocol
complex for C_3



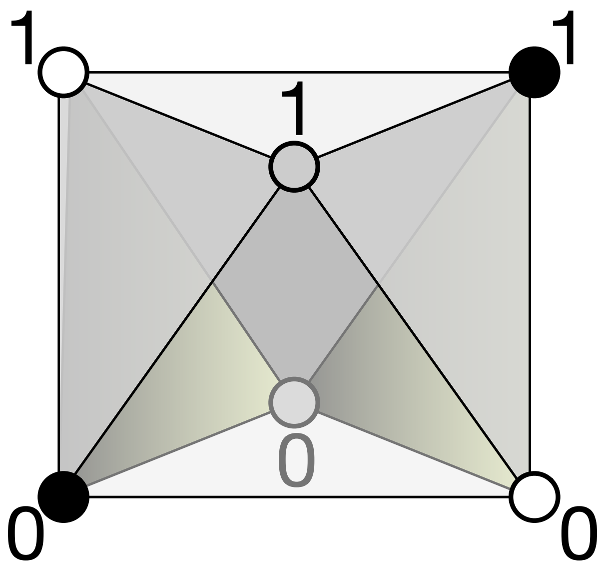




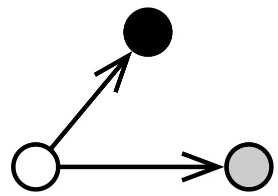
Protocol Complex

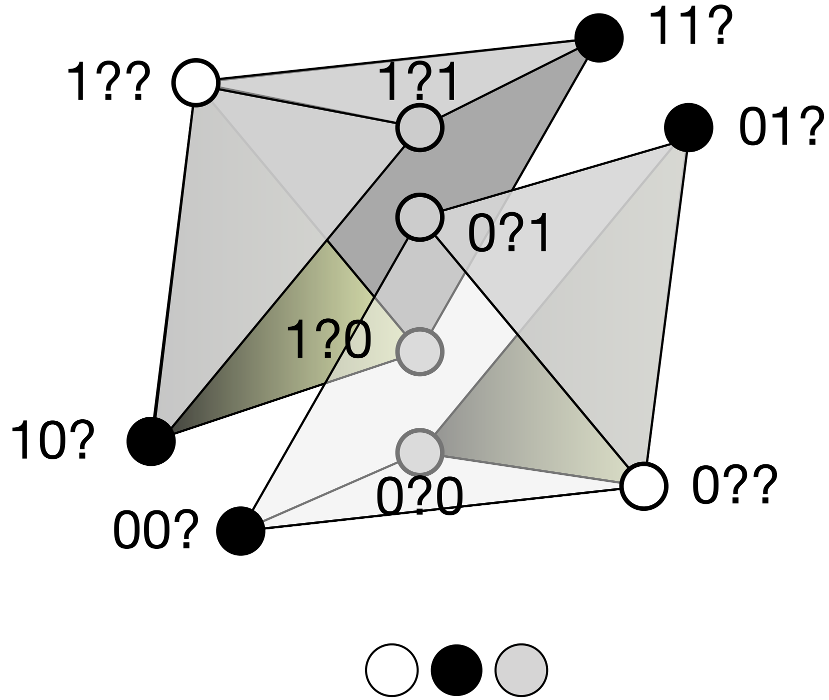
Example 2

Input complex

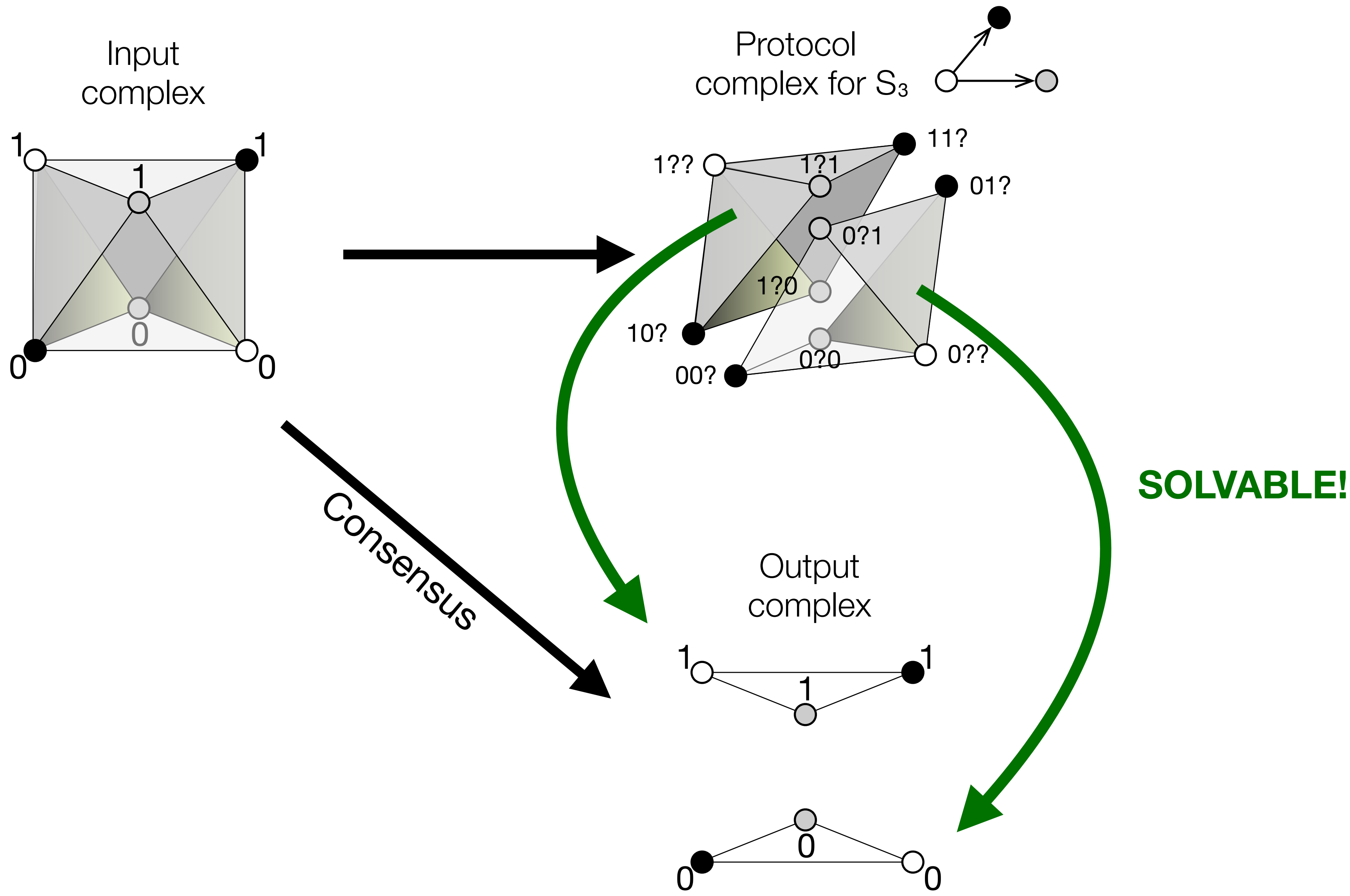


Protocol complex for S_3

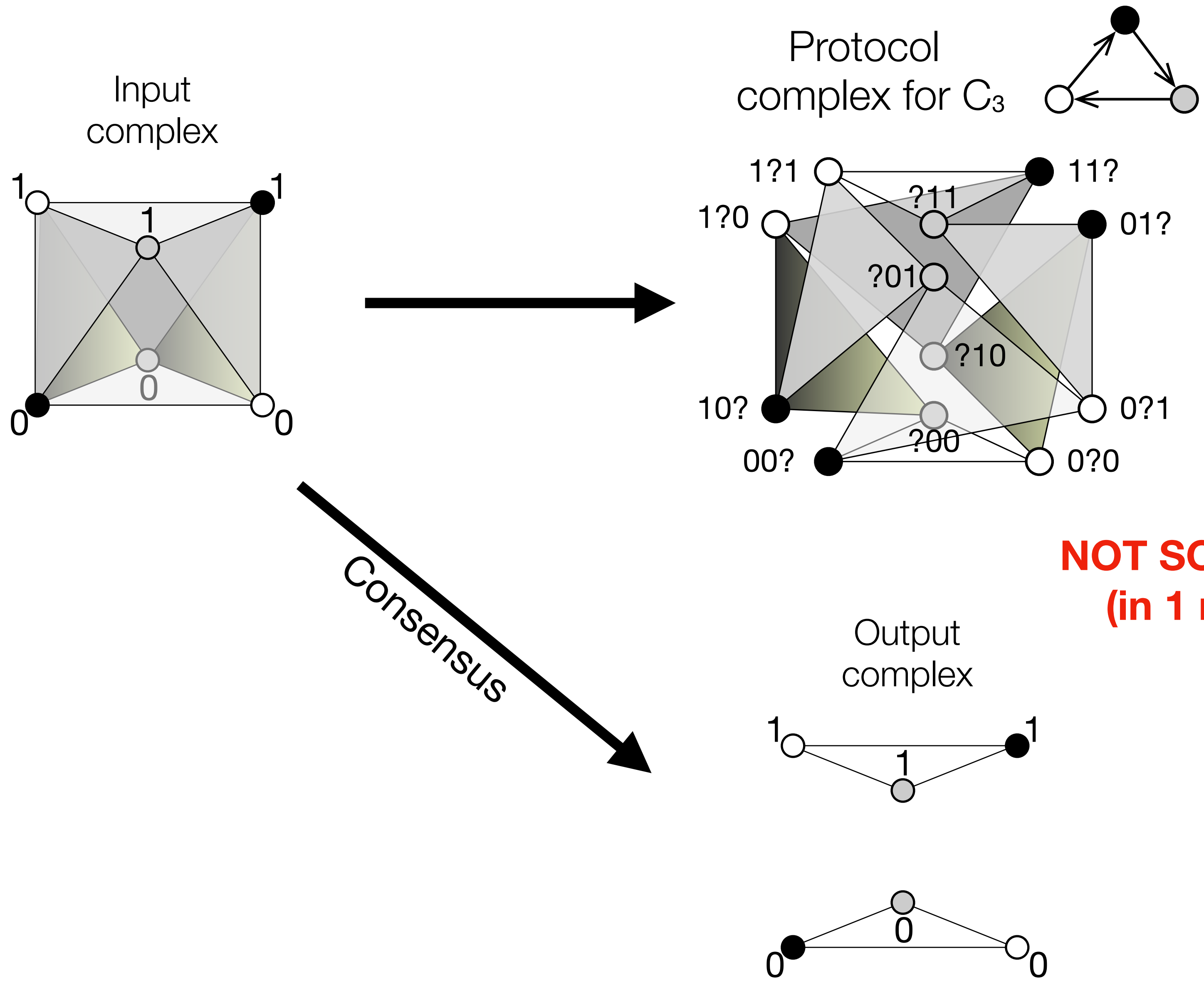




Consensus Solvability



Consensus Solvability



NOT SOLVABLE (in 1 round)

Generalized Brandt's Theorem

Generalization

One can extend Brandt's construction to all round-based iterated models $\mathscr{M}$ supporting full-information protocols:

Generic function $F : \{\text{tasks}\} \to \{\text{tasks}\}$

Theorem [Bastide, F., 2021] For every $t \geq 1$, and every task $\Pi = (\mathscr{I}, \mathscr{O}, \Delta)$, the task $F(\Pi)$ satisfies the following:

- Assume that Π satisfies$(t-1)$-*independence* with respect to $\mathscr{M}$. If Π is solvable in t rounds, then $F(\Pi)$ is solvable in $t-1$ rounds.
- 2. Assume that Π is *locally checkable* in $\mathscr{M}$. If $F(\Pi)$ is solvable in $t-1$ rounds, then Π is solvable in t rounds

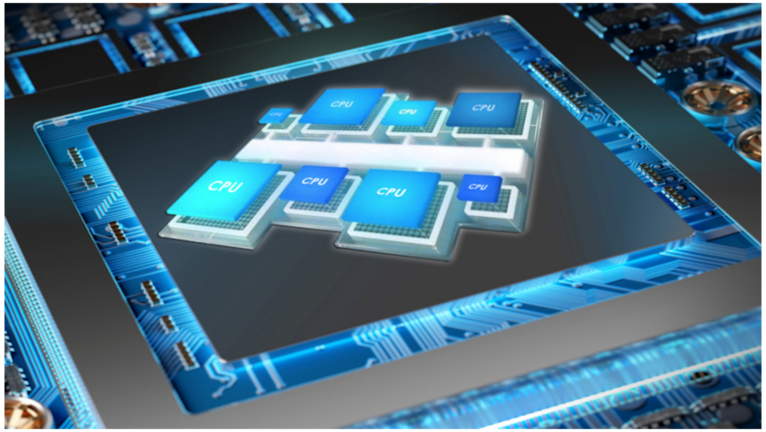
Applications

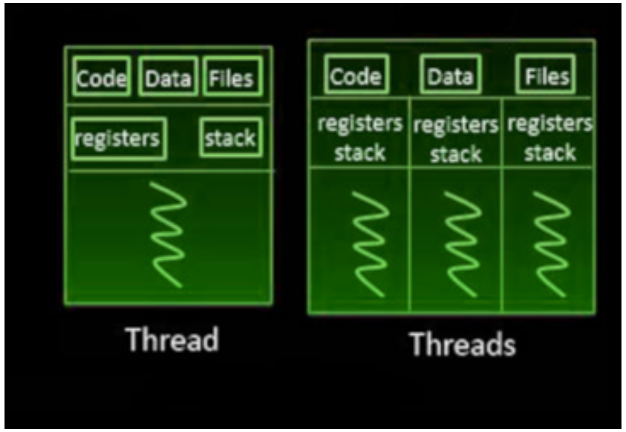
- **Good news:** Extension of Brandt's Theorem to
  - directed graphs, hypergraphs, dynamic networks, etc.
  - graphs including short cycles
  - to 2-process wait-free computing in asynchronous shared-memory: impossibility of consensus and perfect renaming (for 2 processes).

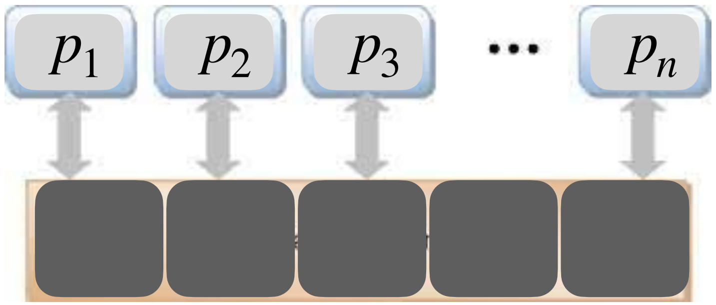
- **Bad news:**
  - Not many models satisfy independence
  - Tasks like consensus are not locally checkable wait-free in the asynchronous shared-memory model.

Wait-Free Computing

Shared Memory Model







Single Writer / Multiple Reader registers

Wait-Free Computing

Code of process $i \in \{1,\ldots,n\}$ with input x_i

$V_i \leftarrow x_i$

For $r = 1$ to t do

  write(V_i) in register $M[i]$

  for $j = 1$ to n do $v_j \leftarrow$ read($M[j]$)

  $V_i \leftarrow (v_1, v_2, \ldots, v_n)$

decide $y_i = f(V_i)$

Atomic operations

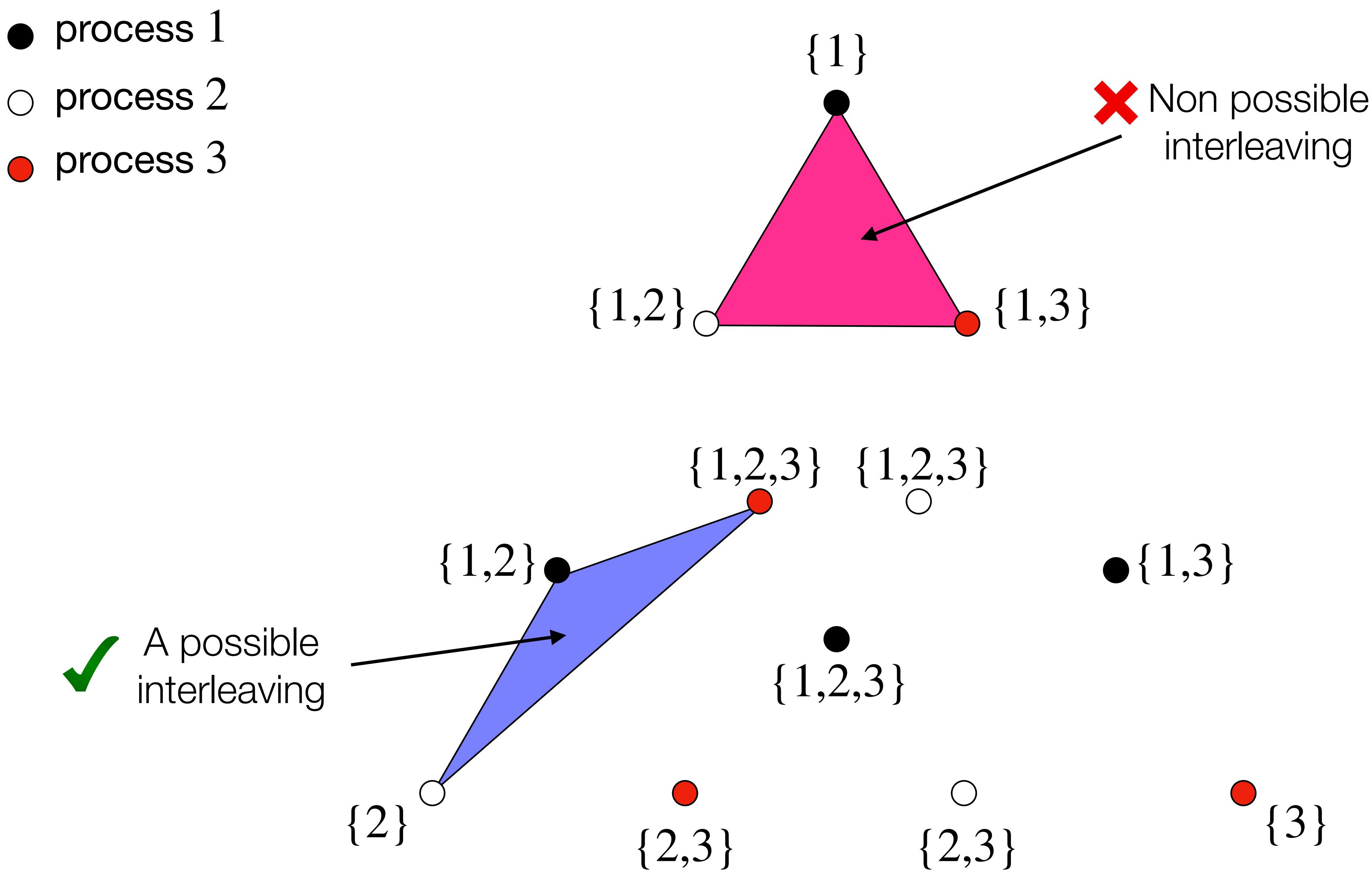
Full Information protocol

Decision function

Read/Write Interleaving

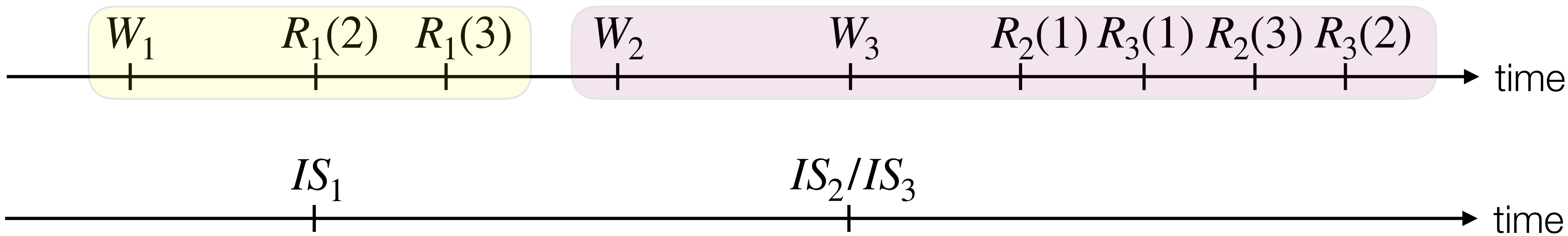
Assume $n = 3$

Read/Write Interleaving

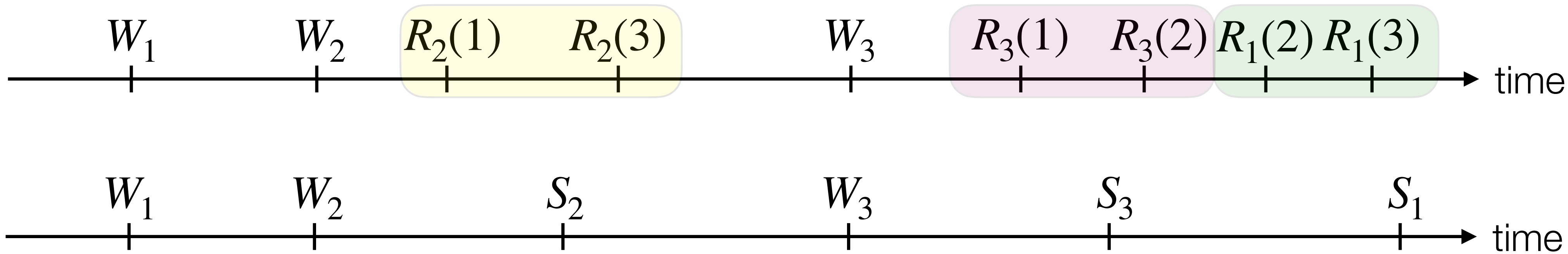


Snapshots and Immediate Snapshots

IMMEDIATE SNAPSHOTS

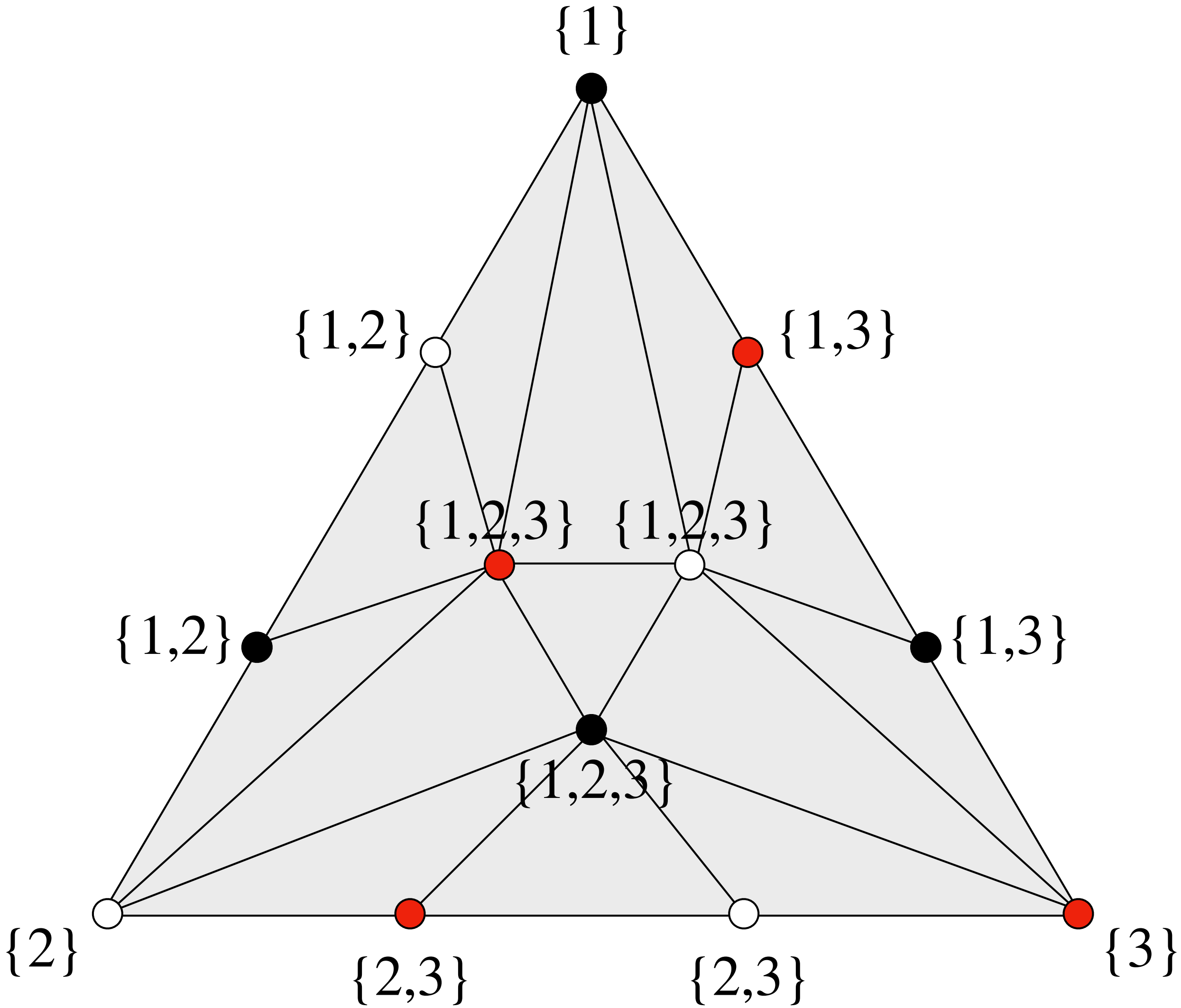


SNAPSHOTS



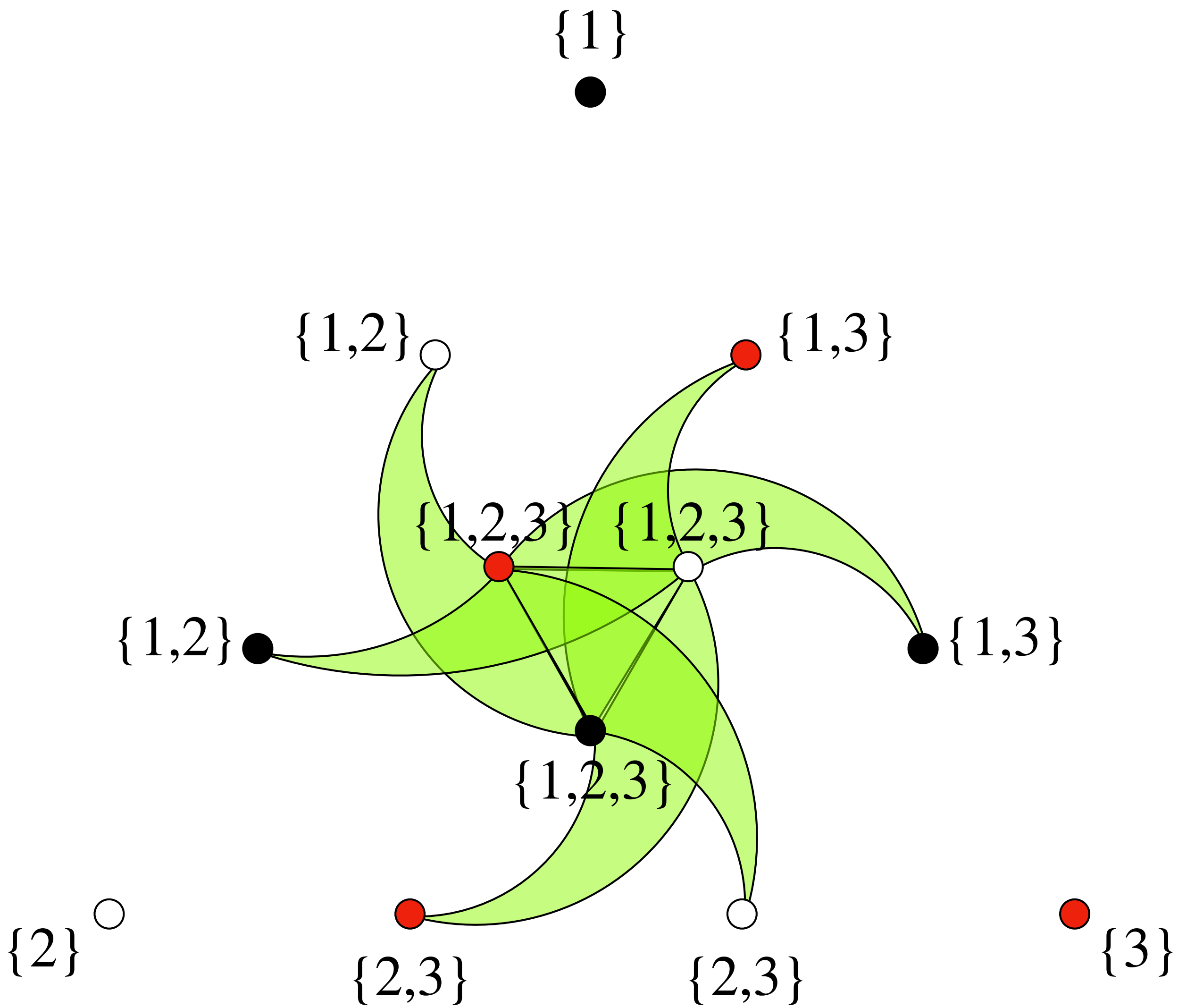
Immediate Snapshots

- ● process 1
- ○ process 2
- ● process 3



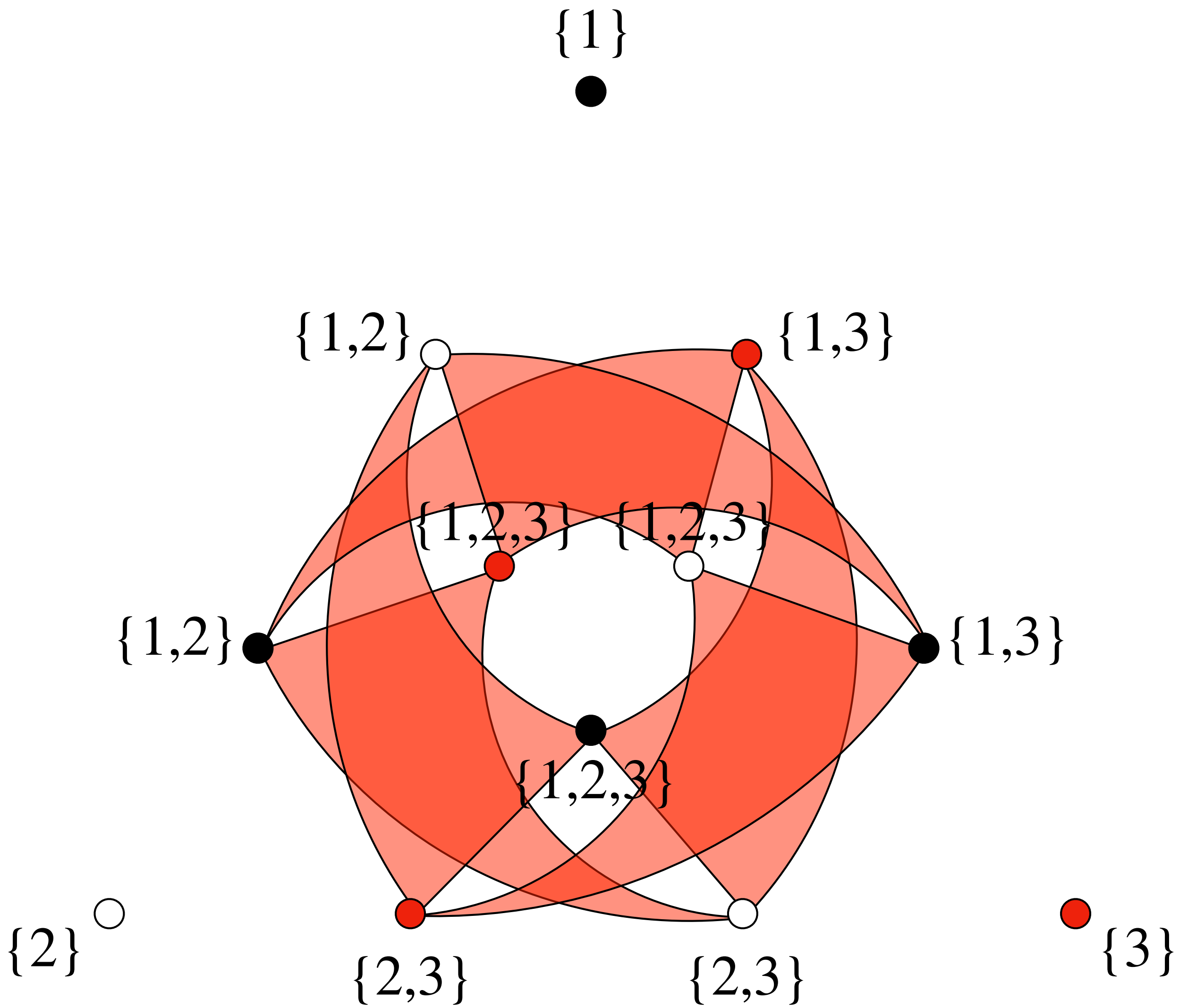
(Non-Immediate) Snapshots

- ● process 1
- ○ process 2
- ● process 3

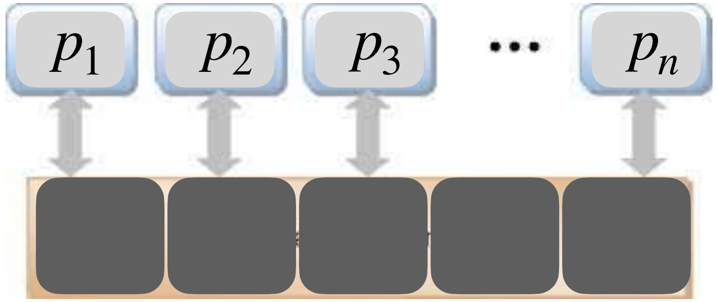


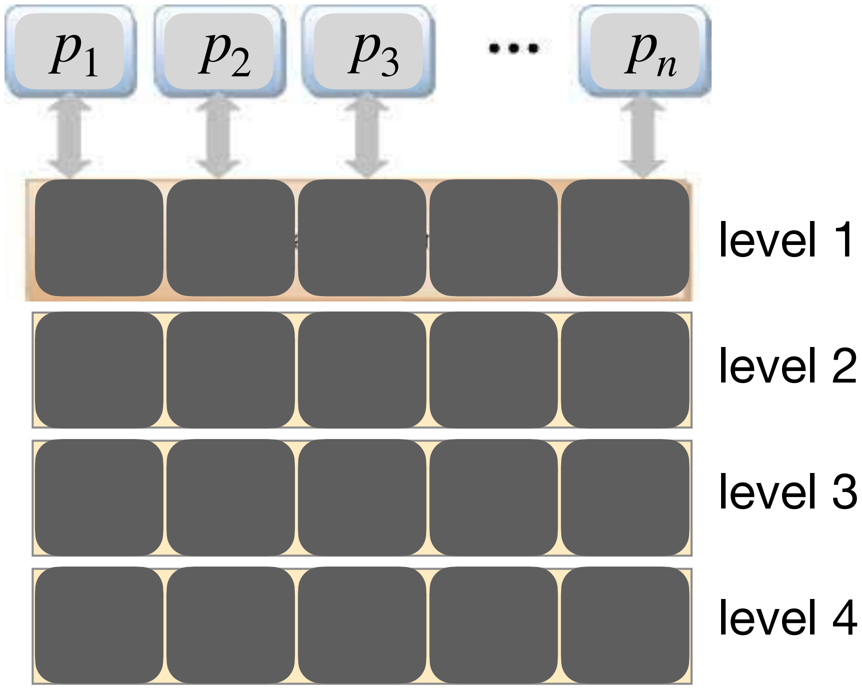
The Rest...

- process 1
- process 2
- process 3



Iterated Model





For every $i = 1,2,\ldots$ the i-th write of each process, as well as all the $n-1$ reads performed after that write are performed in the i-th level of the memory.

Iterated Wait-Free Computing

Code of process $i \in \{1,\ldots,n\}$ with input x_i

$V_i \leftarrow x_i$

For $r = 1$ to t do

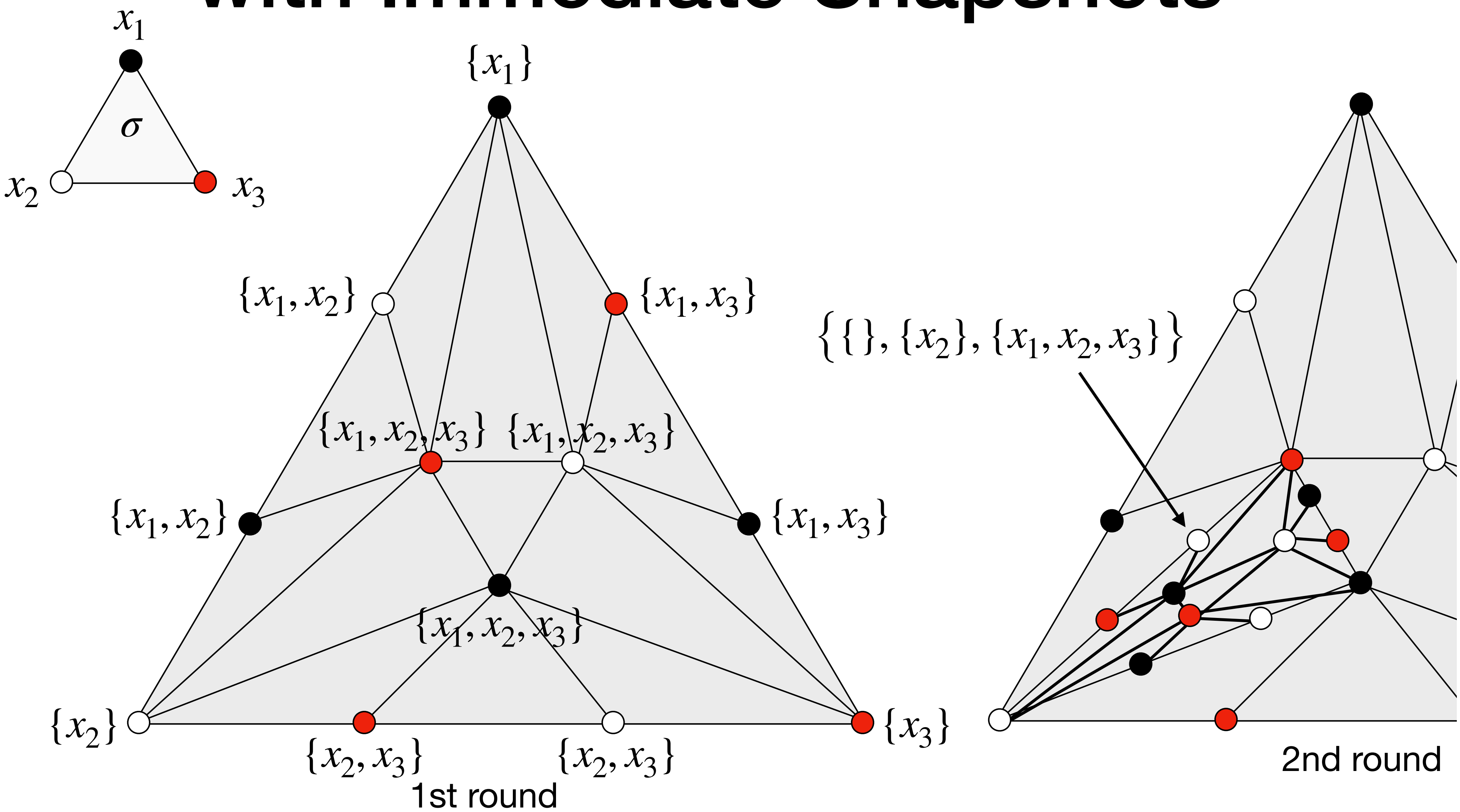
write(V_i) in register $M_r[i]$

for $j = 1$ to n do $v_j \leftarrow$ read($M_r[j]$)

$V_i \leftarrow (v_1, v_2, \ldots, v_n)$

decide $y_i = f(V_i)$

Multi-Round Computation with Immediate Snapshots



Wait-Free Solvability

Lemma For every task Π, Π is solvable wait-free in the asynchronous shared-memory read/write model $\iff$ Π is solvable wait-free in the asynchronous shared-memory IIS model.

Theorem [Herlihy-Shavit, 1999] A task $\Pi = (\mathcal{I}, \mathcal{O}, \Delta)$ is solvable wait-free in the asynchronous shared-memory read/write model if and only if there exists $t \geq 0$ and a simplicial map

$$f : \mathsf{ch}^{(t)}(\mathcal{I}) \to \mathcal{O}$$

t-th chromatic subdivision of $\mathcal{I}$

that agrees with Δ, i.e., for every $\sigma \in \mathcal{I}$,

$$f(\mathsf{ch}^{(t)}(\sigma)) \subseteq \Delta(\sigma).$$

Wait-Free
Speedup Theorem

Intuition

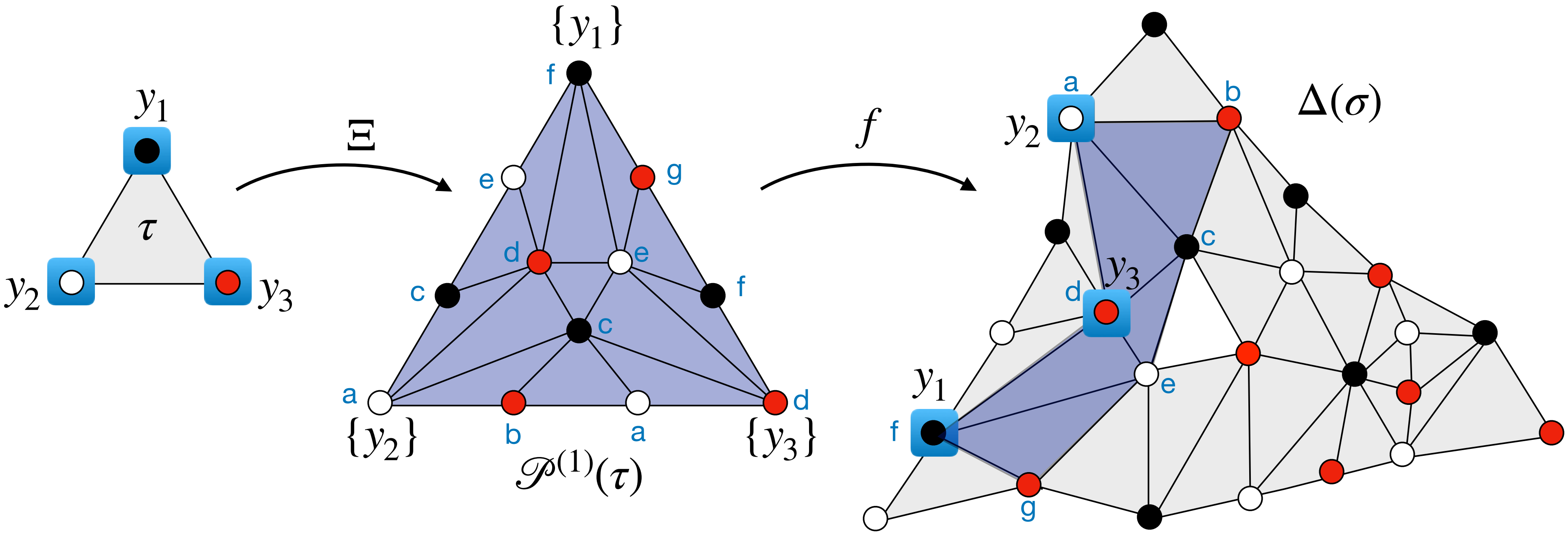
- Let f be a t-round algorithm solving $\Pi = (\mathcal{I}, \mathcal{O}, \Delta)$ in the wait-free IIS model

- What can be done in $t-1$ rounds?

- Every process starting with input $x \in V(\mathcal{I})$:
  1. performs $t-1$ rounds: state s
  2. assumes running solo during the t-th round: state $\{s\}$
  3. outputs $y = f(\{s\}) \in V(\mathcal{O})$

- What properties satisfy these outputs?

These Outputs are Close to Each Other



Local Tasks

- Let $\Pi = (\mathcal{I}, \mathcal{O}, \Delta)$ be a task
- Let $\sigma \in \mathcal{I}$
- Let $\tau \subseteq \Delta(\sigma)$ be a chromatic set with name(τ) $=$ name(σ)
- Local task $\Pi_{\tau,\sigma} = (\tau, \Delta(\sigma), \Delta_{\tau,\sigma})$ where
  - $\Delta_{\tau,\sigma}(\tau') = \tau'$ if $|\tau'| = 1$
  - $\Delta_{\tau,\sigma}(\tau') =$ proj$_{name(\tau')}(\Delta(\sigma))$ if $|\tau'| > 1$

Closure Tasks

Definition The closure of a task $\Pi = (\mathcal{I}, \mathcal{O}, \Delta)$ is the task

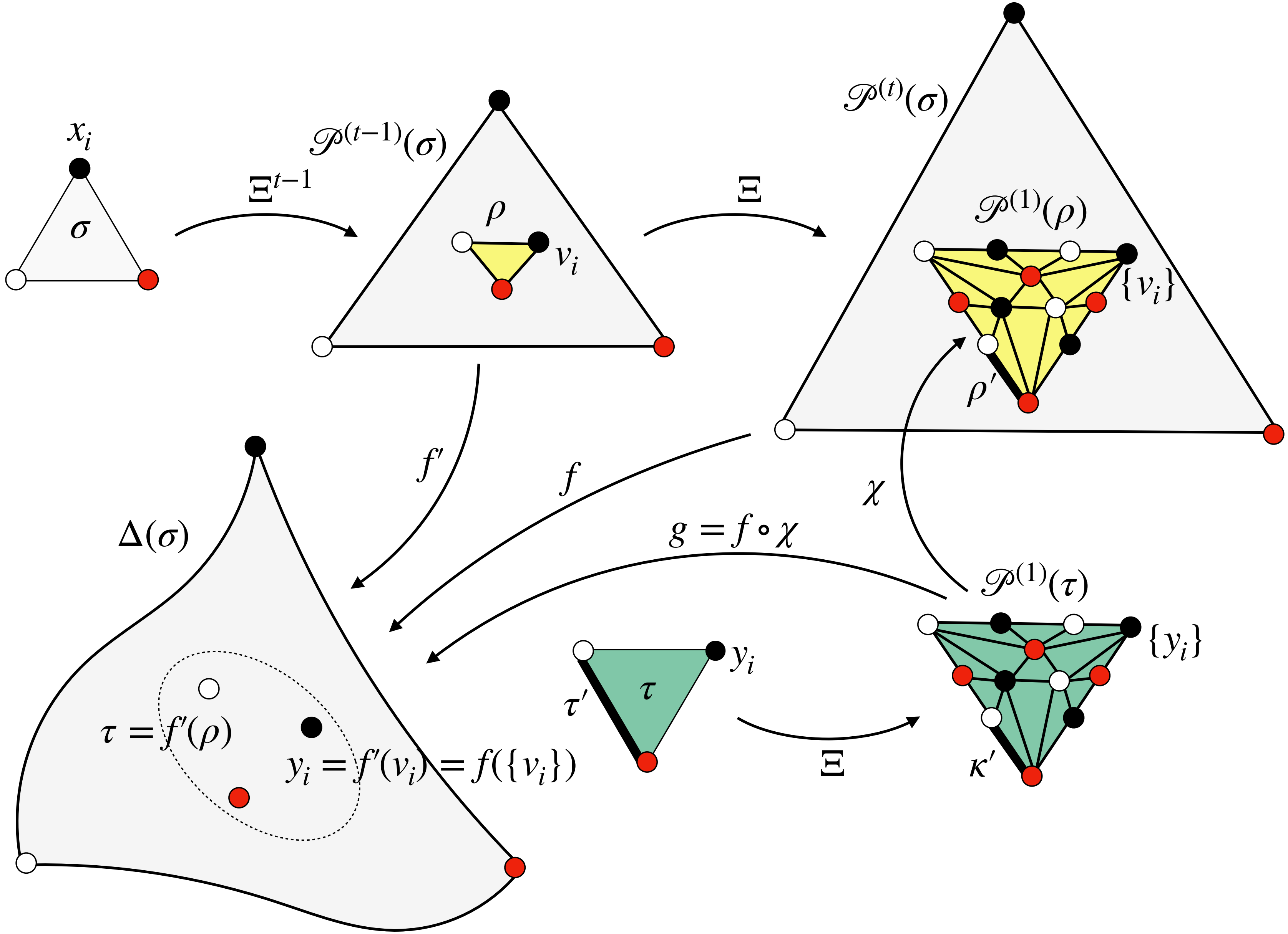
$$\text{closure}(\Pi) = (\mathcal{I}, \mathcal{O}', \Delta')$$

where $V(\mathcal{O}') = V(\mathcal{O})$ and, for every $\sigma \in \mathcal{I}$ and $\tau \subseteq V(\mathcal{O})$, we set $\tau \in \Delta'(\sigma)$ if

1. $\text{name}(\tau) = \text{name}(\sigma)$ and $\tau \subseteq V(\Delta(\sigma))$
2. the local task $\Pi_{\tau,\sigma}$ is solvable in 1 round.

Theorem [F., Paz, Rajsbaum, 2022] For every $t \geq 1$, and every task $\Pi = (\mathcal{I}, \mathcal{O}, \Delta)$, if Π is solvable in t rounds then $\text{closure}(\Pi)$ is solvable in $t - 1$ rounds.

Proof



x_i
σ
Ξ^{t-1}
$\mathscr{P}^{(t-1)}(\sigma)$
ρ
v_i
Ξ
$\mathscr{P}^{(t)}(\sigma)$
$\mathscr{P}^{(1)}(\rho)$
$\{v_i\}$
ρ'
f'
f
χ
$g = f \circ \chi$
$\Delta(\sigma)$
$\mathscr{P}^{(1)}(\tau)$
$\{y_i\}$
y_i
τ
τ'
$\tau = f'(\rho)$
$y_i = f'(v_i) = f(\{v_i\})$
Ξ
κ'

Applications

- closure(*consensus*) = *consensus* $\Longrightarrow$ impossibility of *consensus*.

- closure(ϵ-*agreement*) = (2ϵ)-*agreement* $\Longrightarrow$ lower bound $\lceil \log_2 1/\epsilon \rceil$ rounds for ϵ-*agreement*.

- extension to models including test&set and binary-consensus objects.

- However, closure(*set-agreement*) is trivial, i.e., can be solved in zero rounds.

Wrap Up

Conclusion and Open Problems

- Algebraic topology bridges the different models of distributed computing.

- Which tasks have non-trivial closures?

- Is there an if-and-only-if speedup theorem for asynchronous wait-free computing?

- Which (full information) models allow for the design of (useful) speedup theorem? E.g., what about t-resilient models?