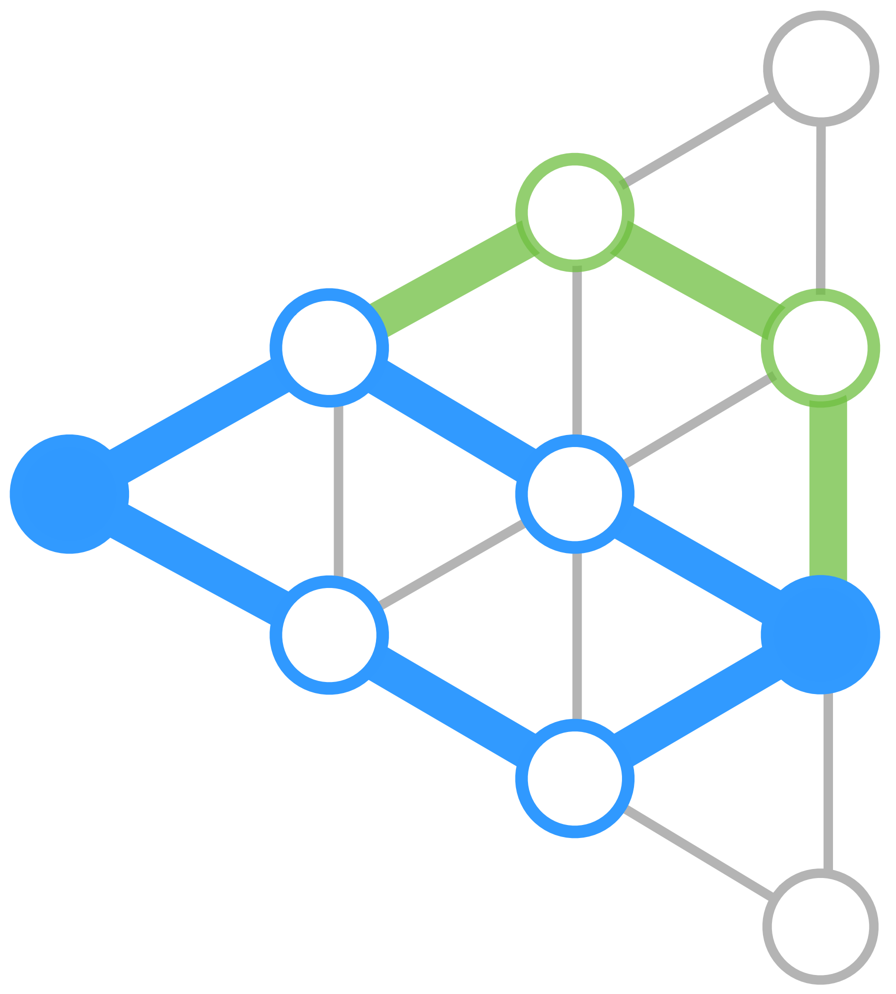


# Approximate agreement on graphs revisited

**ADGA 2023**

**Joel Rybicki**
Humboldt University of Berlin

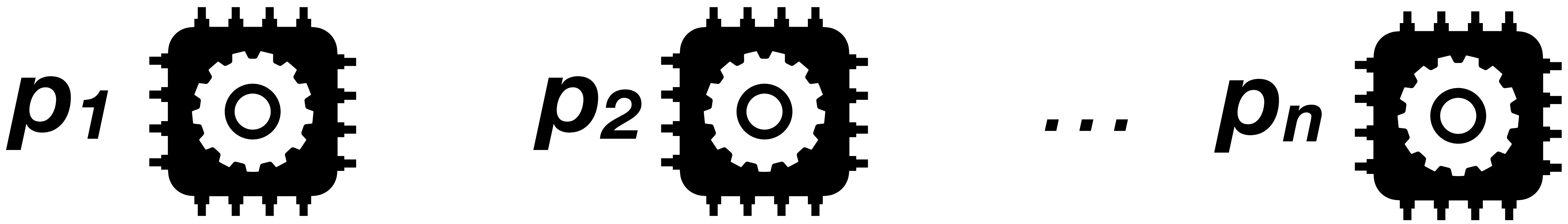
October 9, 2023

**Based on joint work with:**
Dan Alistarh
Faith Ellen
Thomas Nowak

# Outline

1. Approximate agreement on graphs
2. Wait-free shared memory algorithms
3. Impossibility results
4. Open problems

# Distributed agreement tasks



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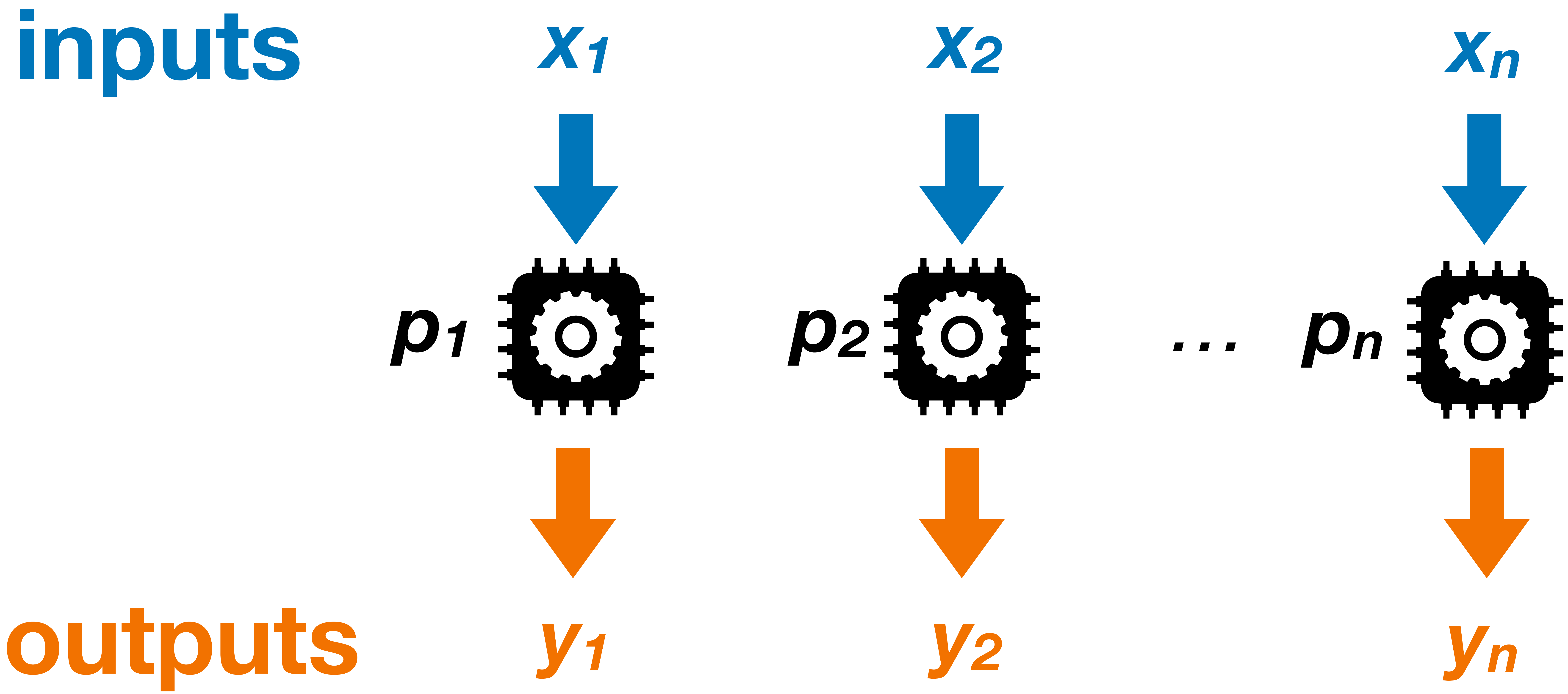
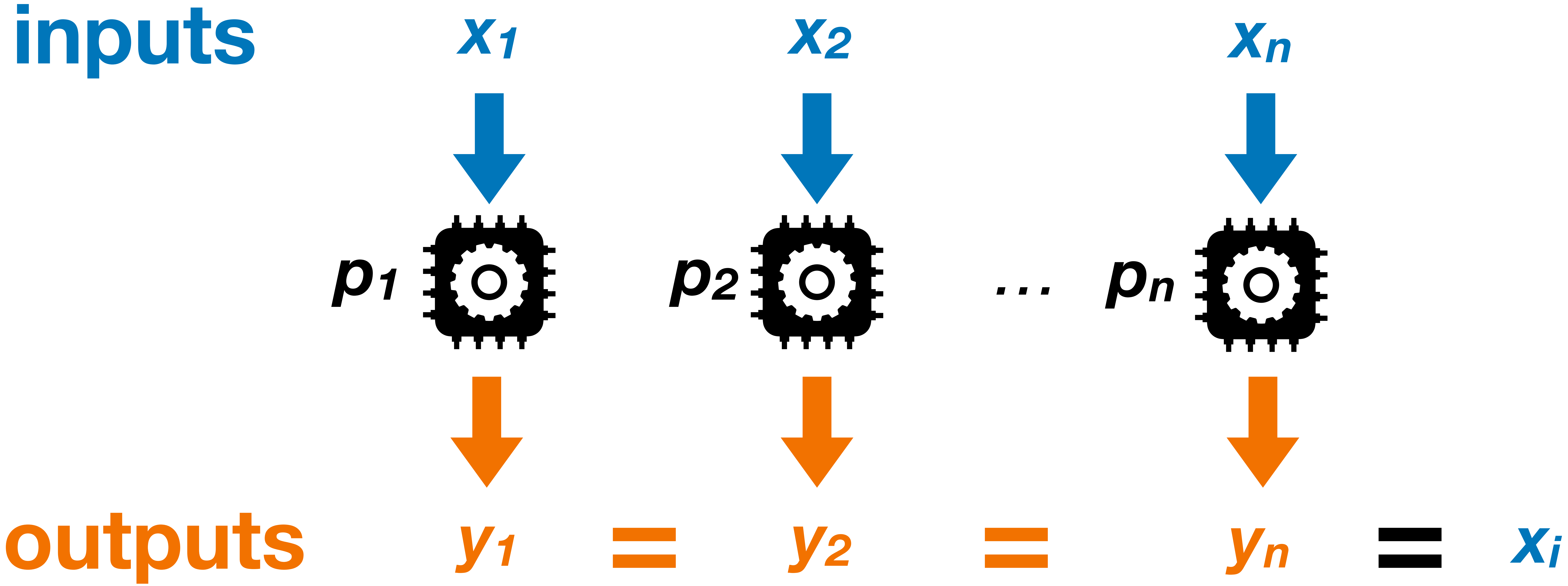


# Distributed agreement tasks:
consensus



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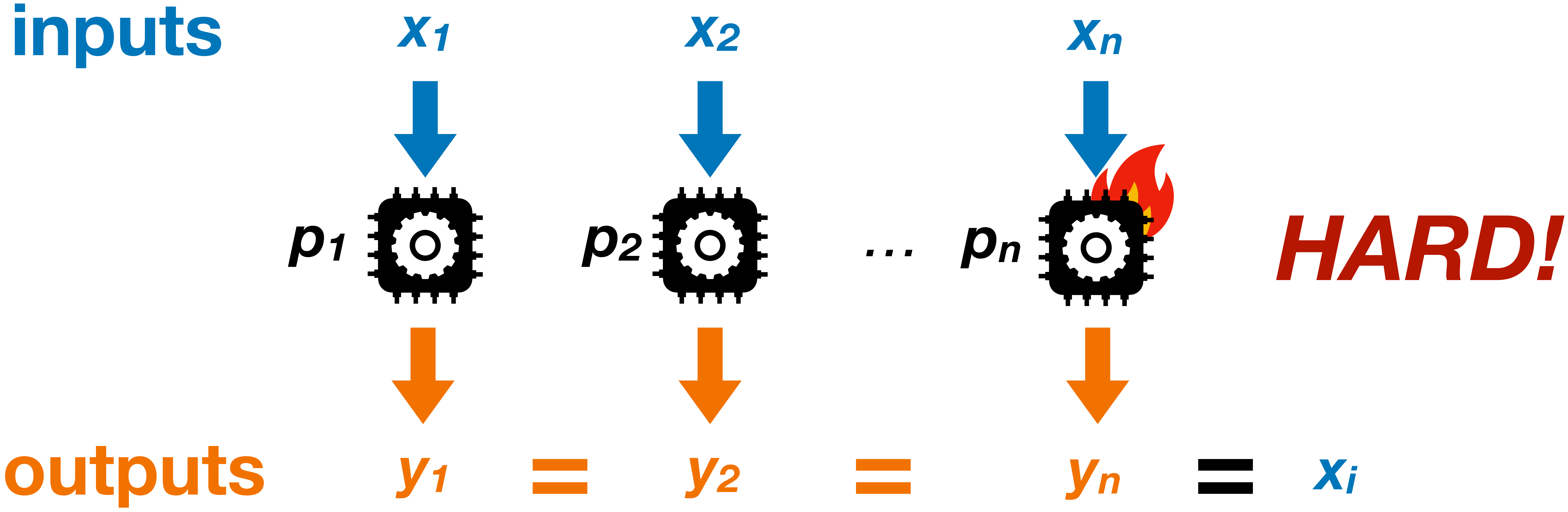
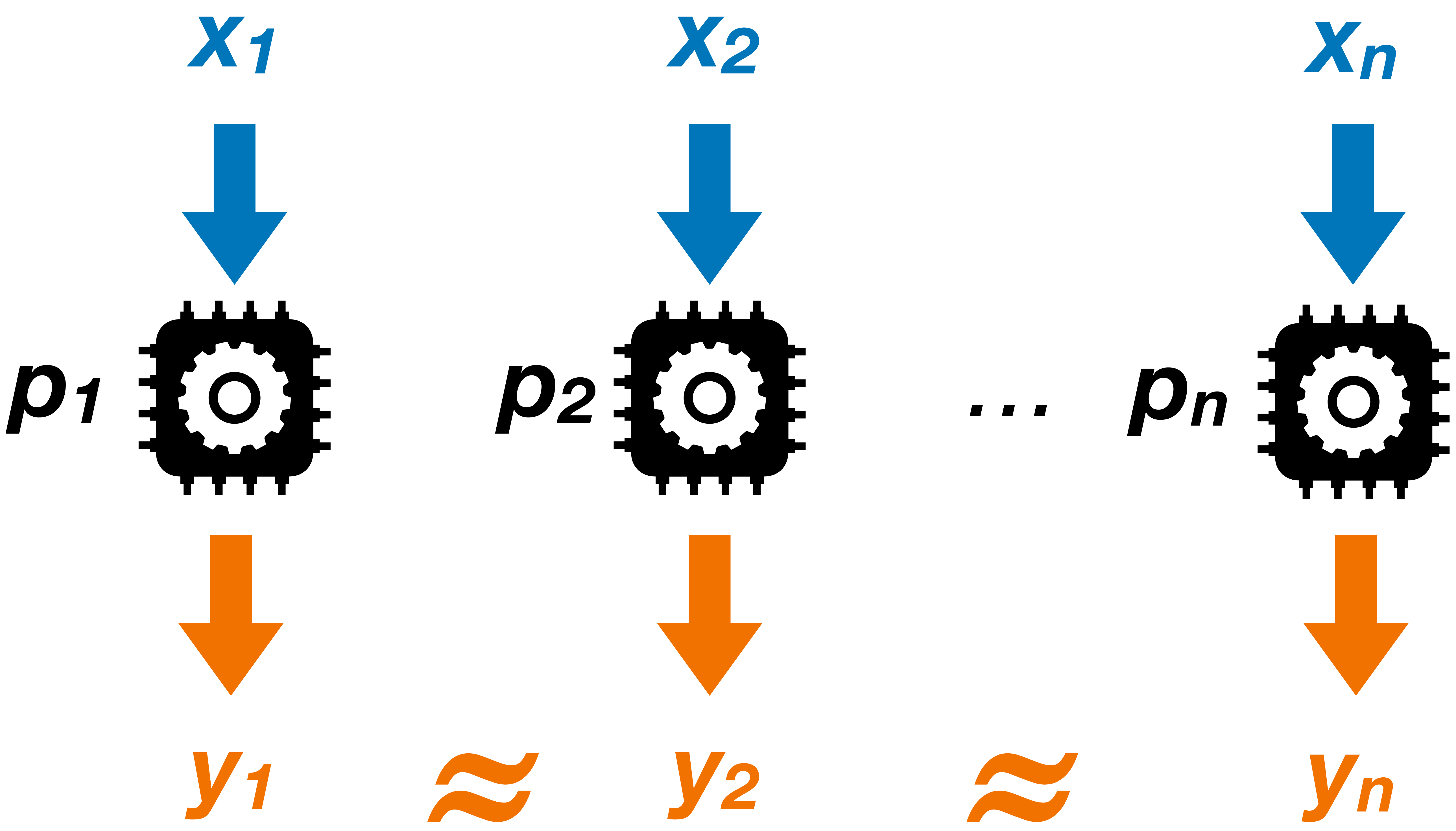
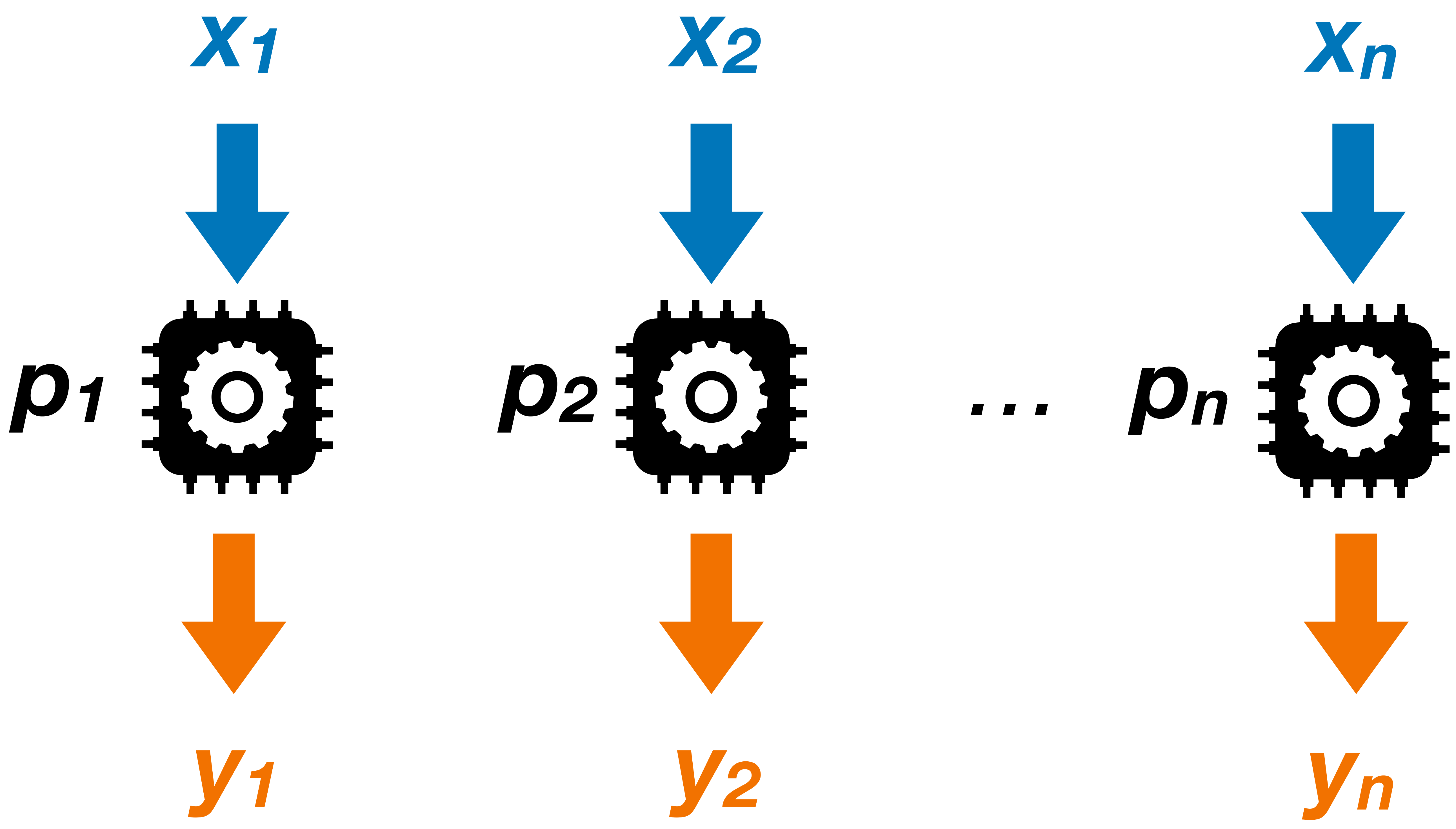



e.g., **Fischer, Lynch, Paterson (1985)**

# **Distributed agreement tasks:**
approximate agreement

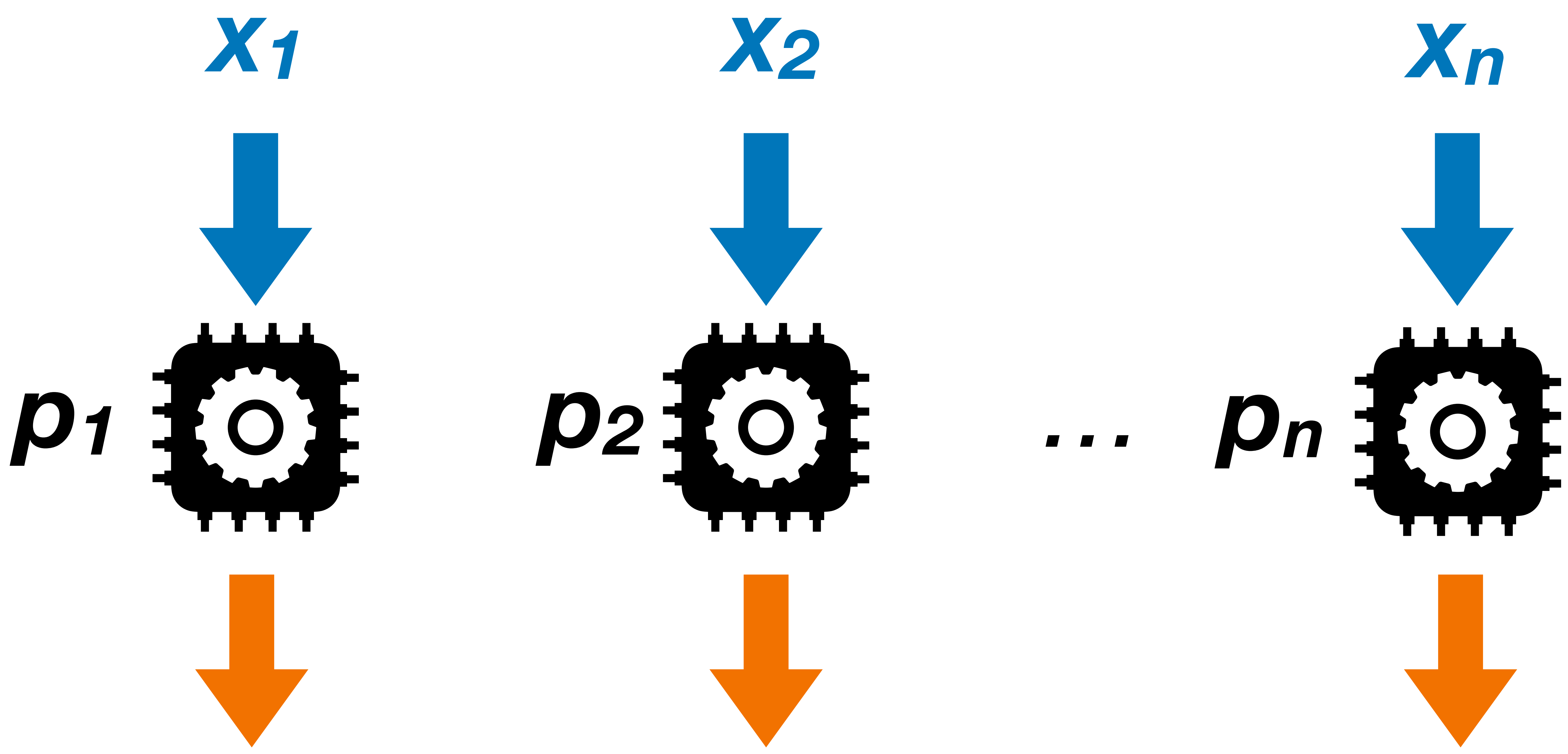


# Distributed agreement tasks:
approximate agreement over real numbers




distance( $y_i$ , $y_j$ ) $\leq \varepsilon$

# **Distributed agreement tasks:**
## approximate agreement over real numbers




$$\min x_i \quad \leq \quad y_1 \qquad y_2 \qquad y_n \quad \leq \quad \max x_i$$

$$\mathrm{distance}(y_i, y_j) \leq \varepsilon$$

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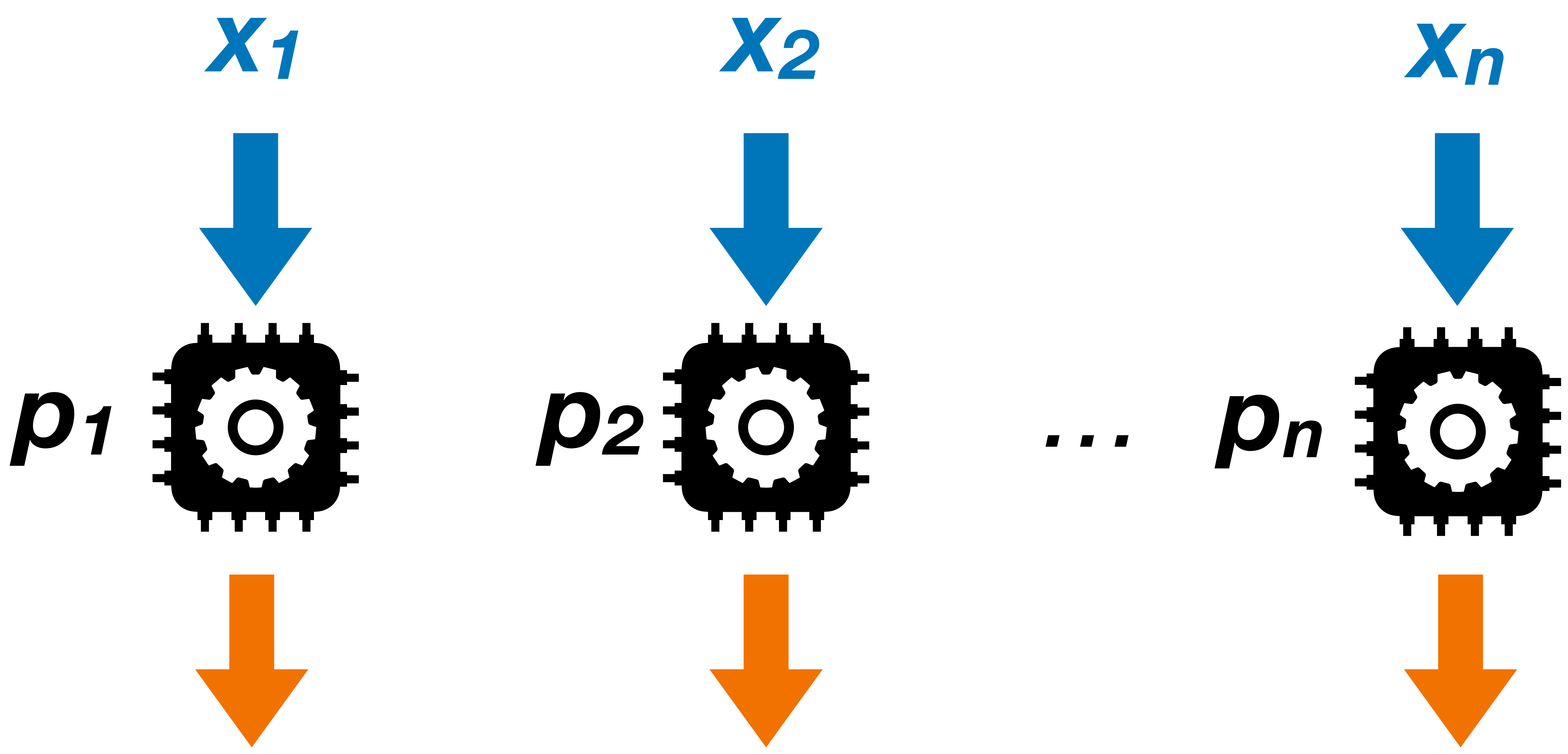



**EASY!**

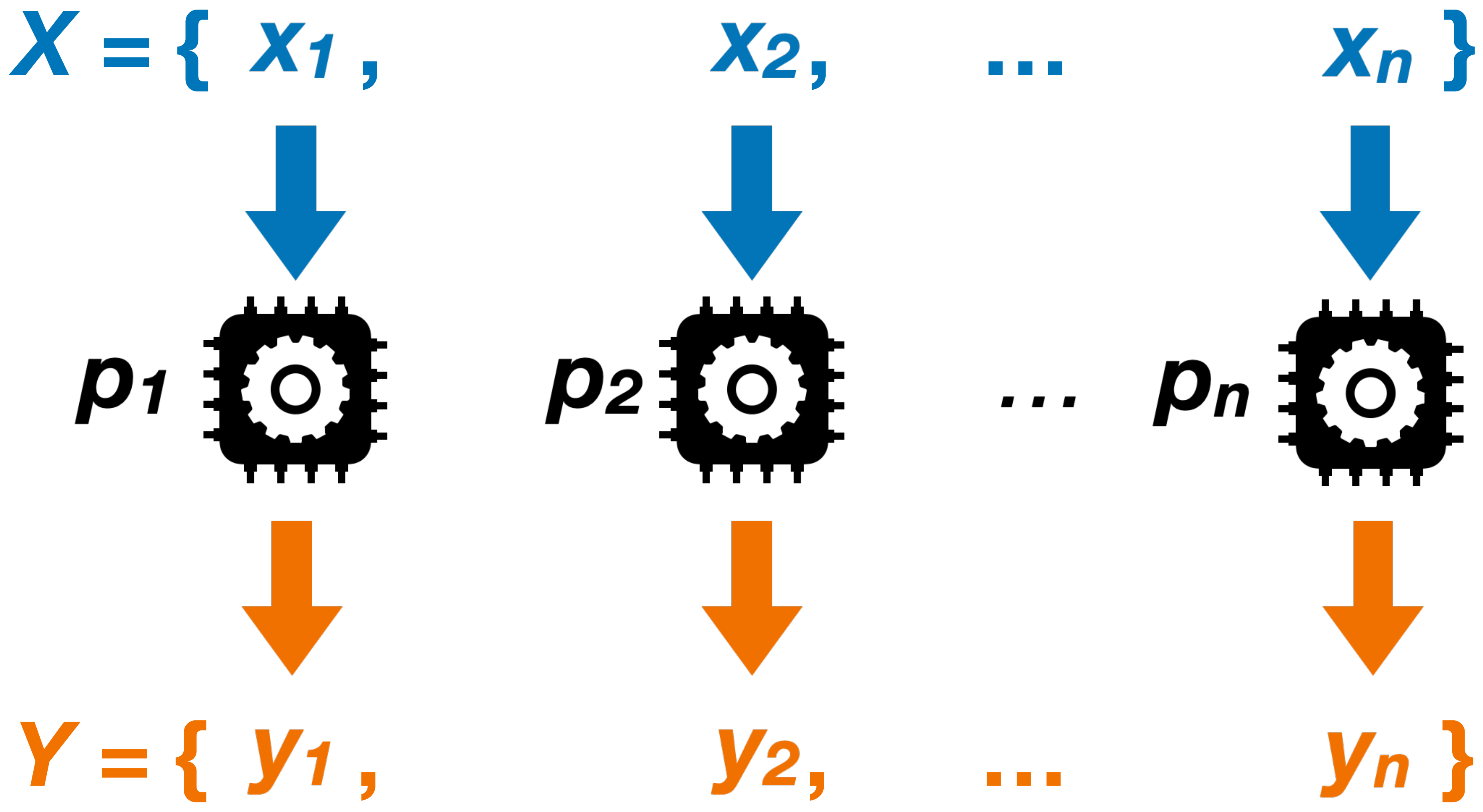
$$\min x_i \le y_1 \quad y_2 \quad \dots \quad y_n \le \max x_i$$

$$\text{distance}(y_i, y_j) \le \varepsilon$$

e.g., **Dolev, Lynch, Pinter, Stark, Weihl (1986)**

# Approximate agreement tasks

A fixed set $\boldsymbol{V}$ of values



# Approximate agreement tasks

A fixed set $\boldsymbol{V}$ of values

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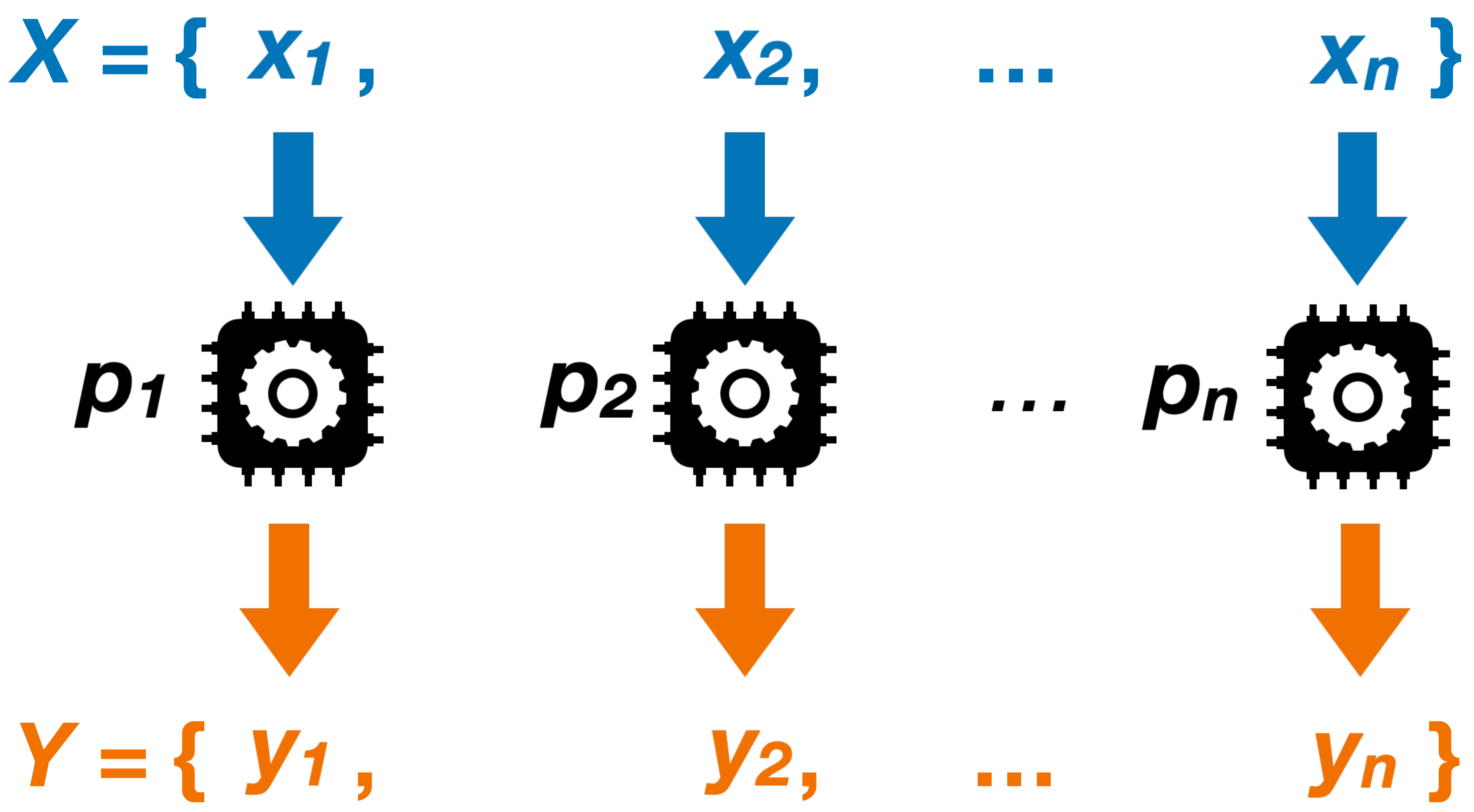
distance( $y_i, y_j$ ) $\leq \varepsilon$ for any $y_i$, $y_j$



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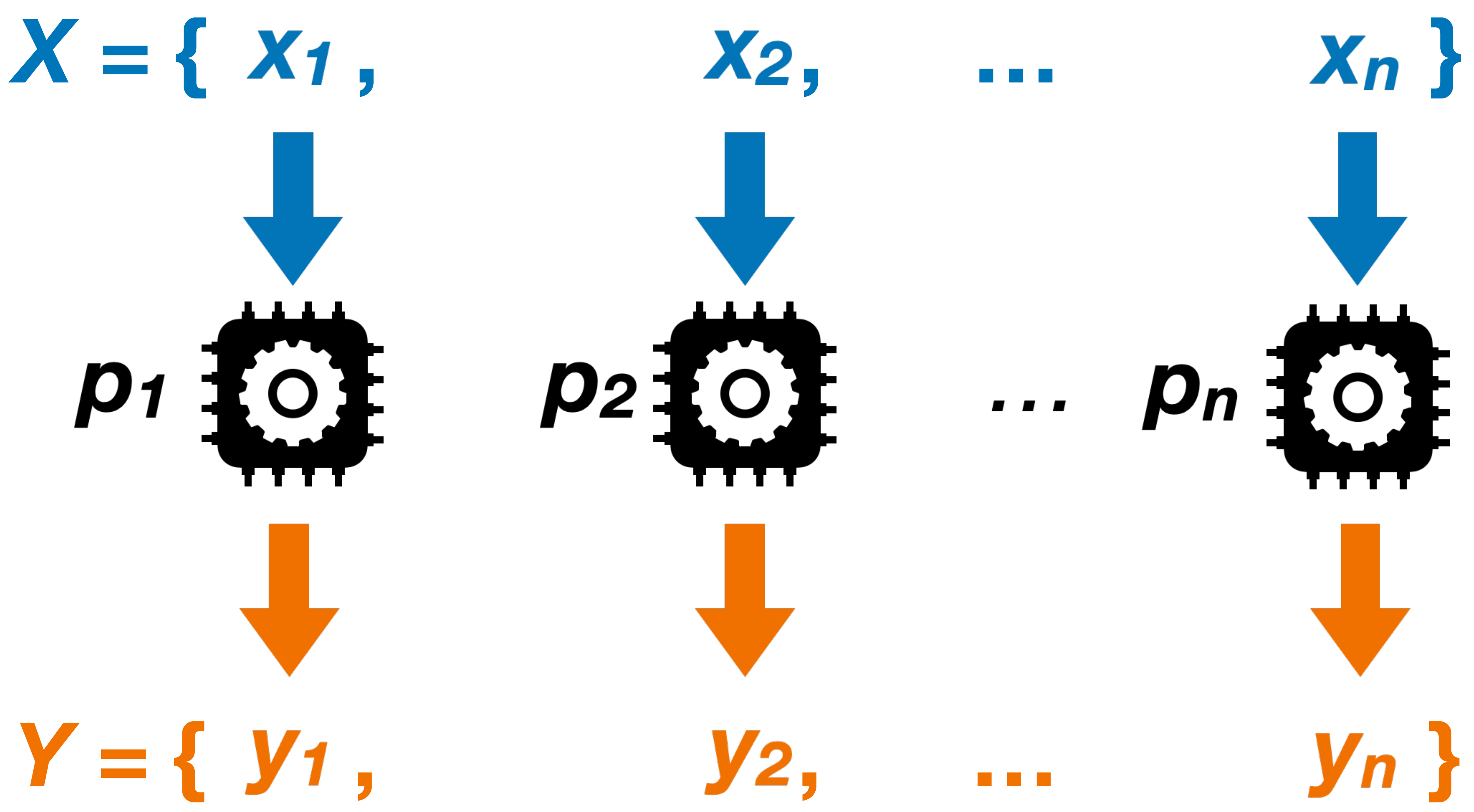

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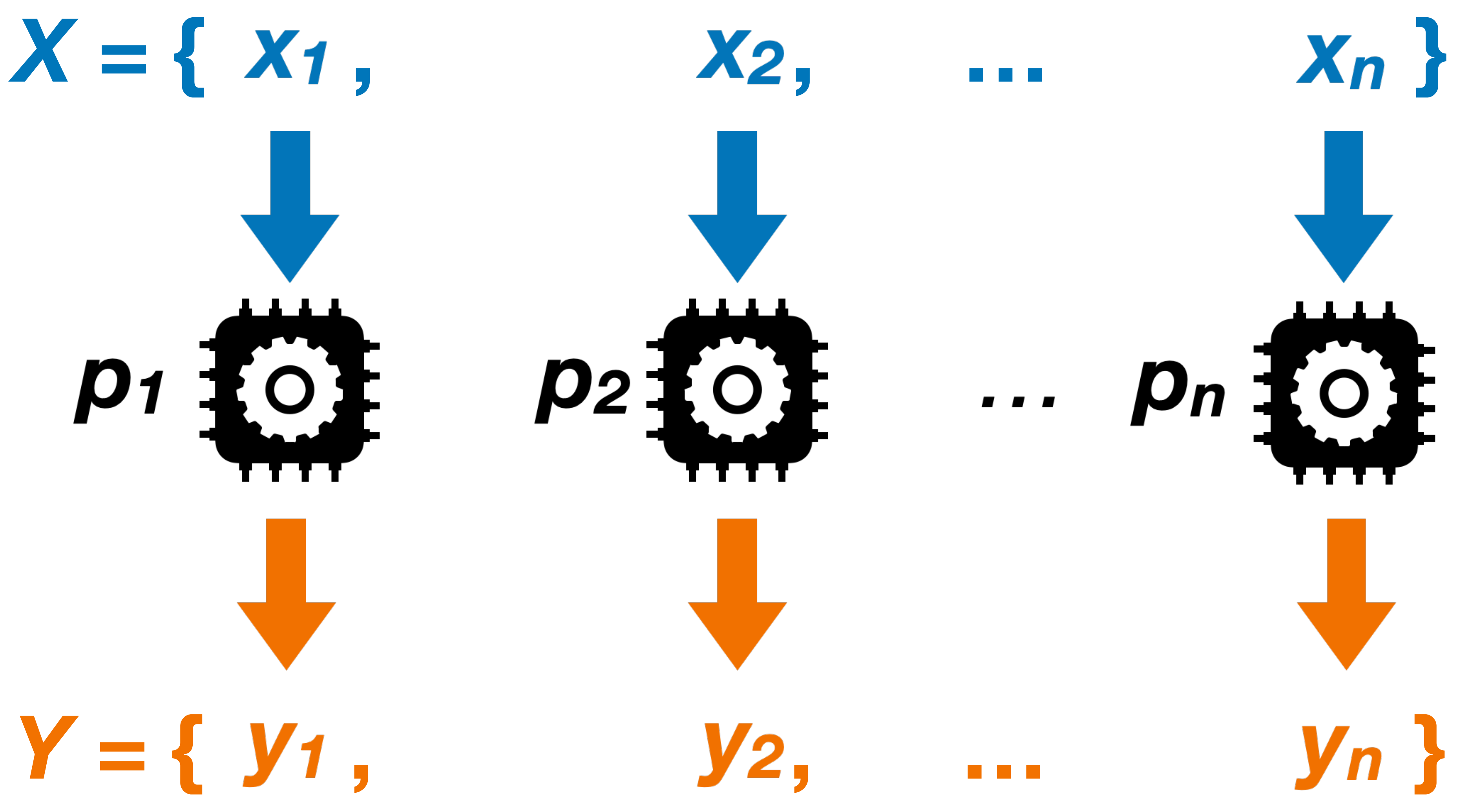


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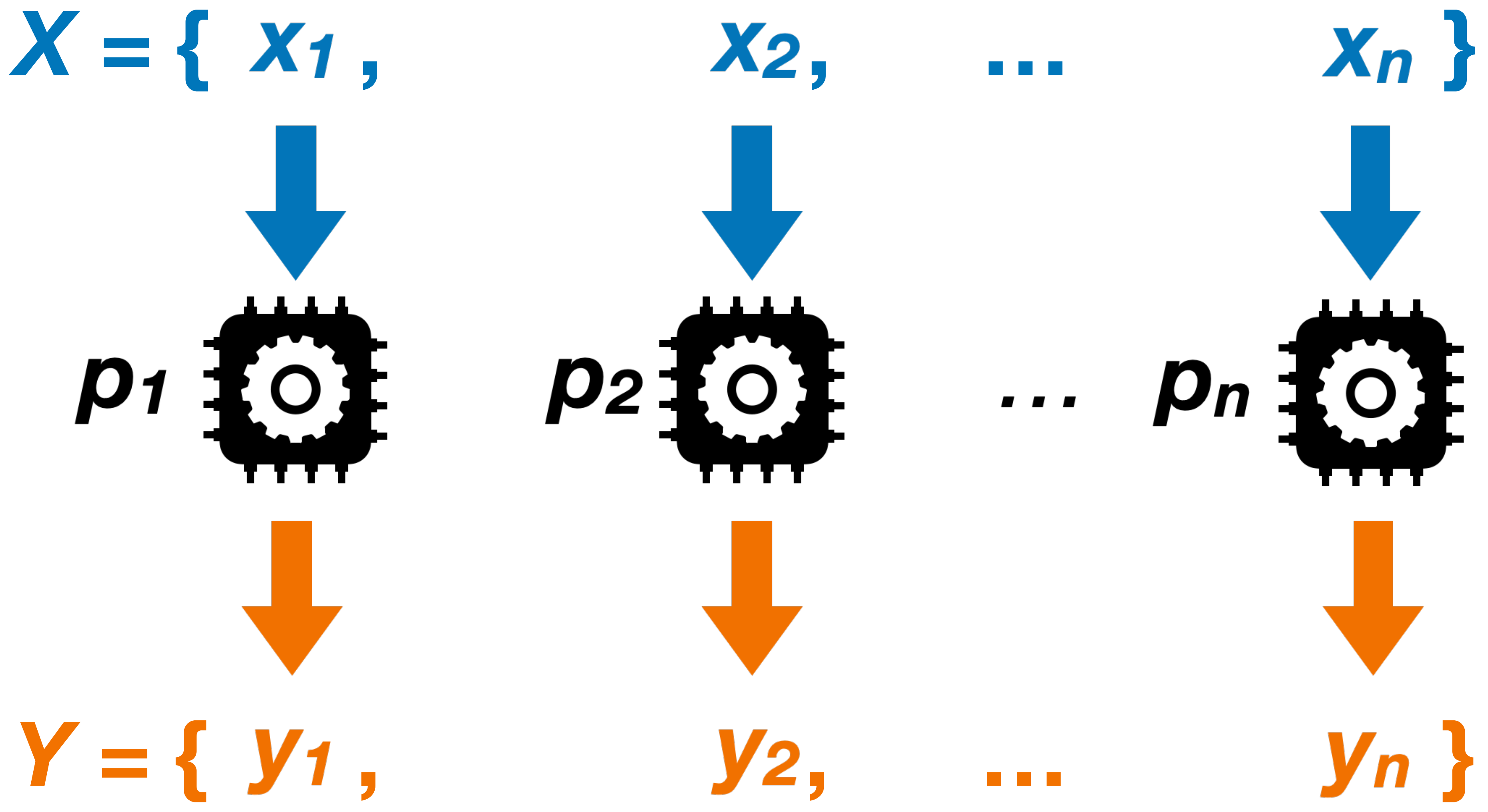


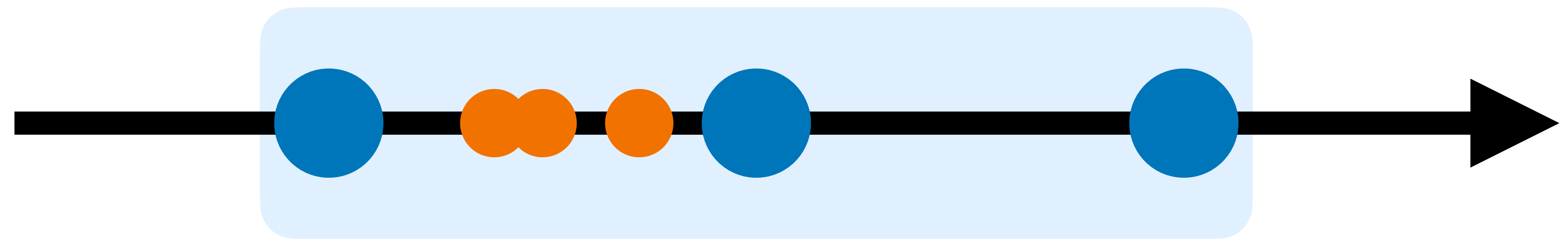

e.g., **Dolev, Lynch, Pinter, Stark, Weihl (1986)**
**Abraham, Amit, Dolev (2004)**
approximate agreement over reals

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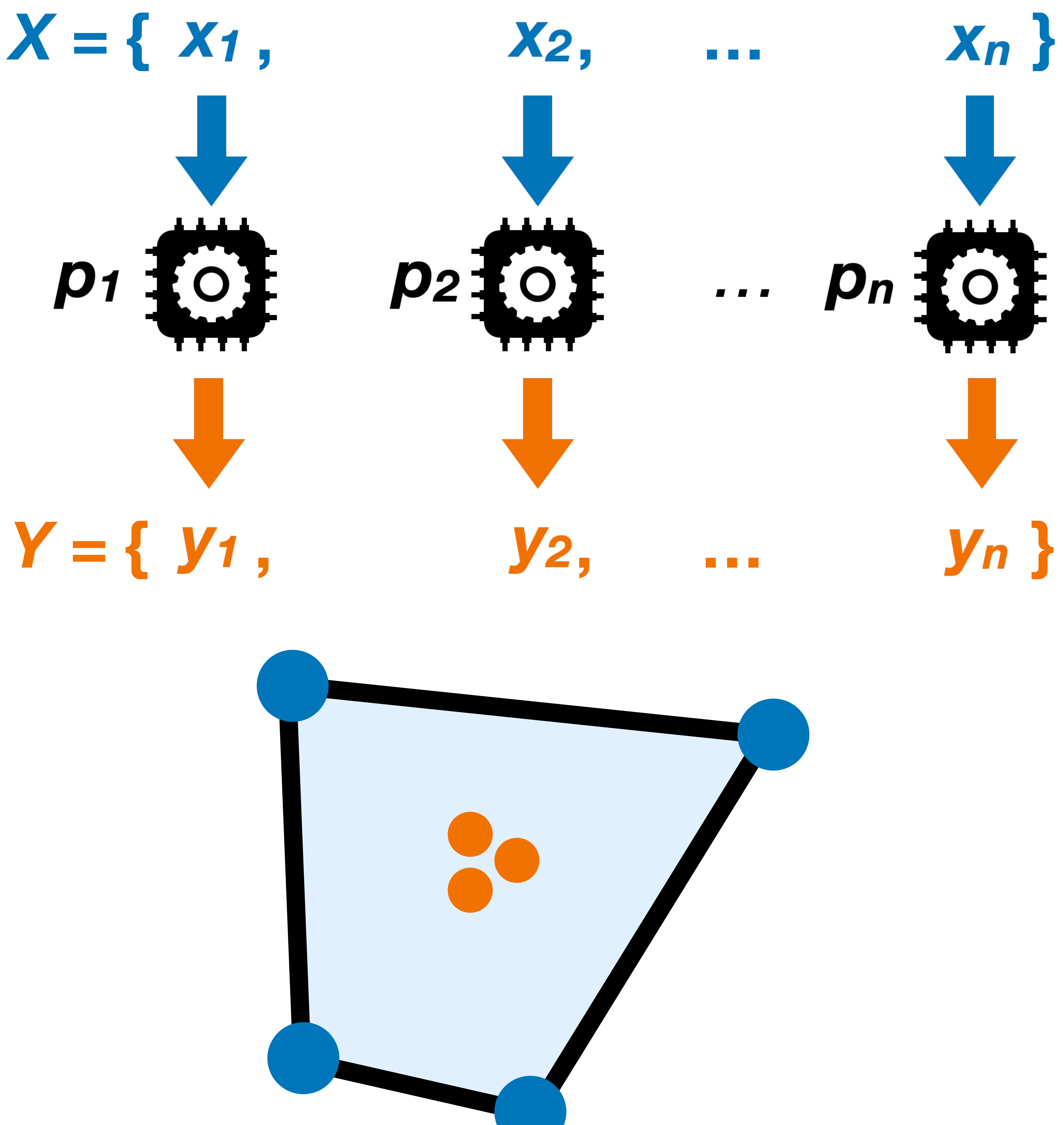


e.g., **Mendes, Herlihy, Vaidya, Garg (2015)**
multidimensional approximate agreement

# Approximate agreement tasks

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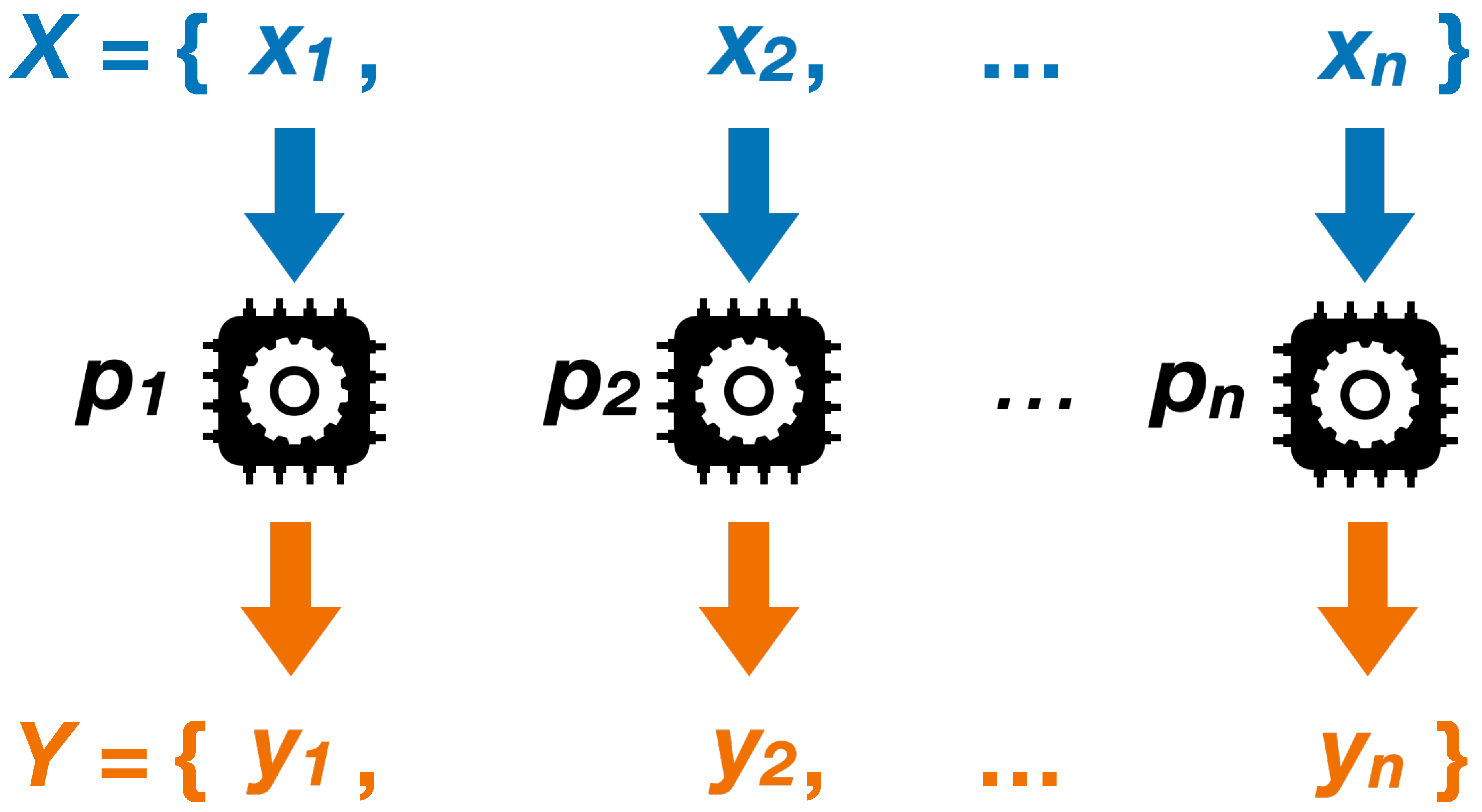
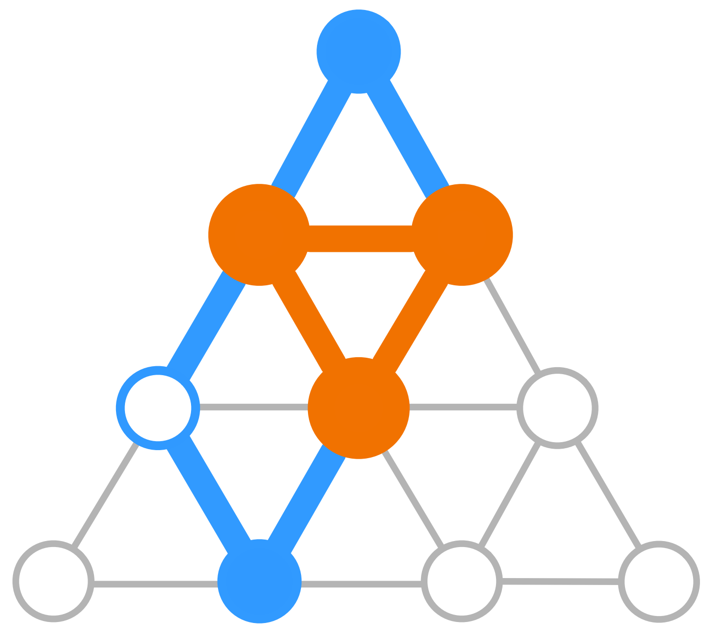
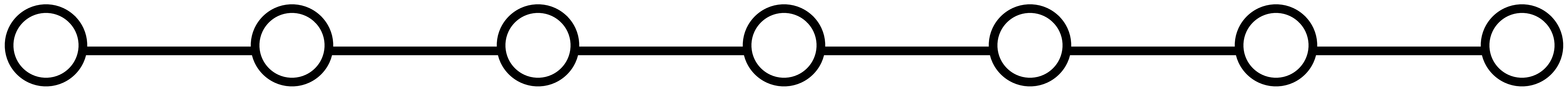


e.g. **Nowak and Rybicki (DISC 2019)**
approximate agreement on graphs and lattices

# Approximate agreement on graphs

**Inputs** and **outputs** are vertices of a fixed graph ***G***



***G***

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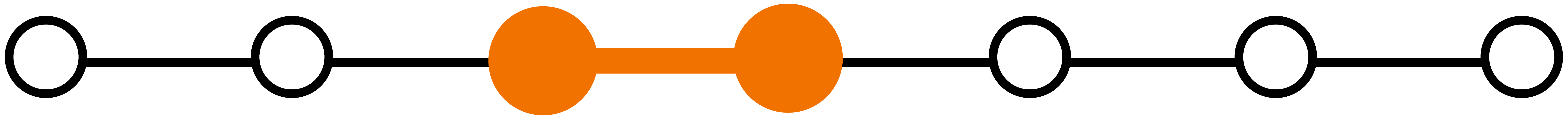

***G***

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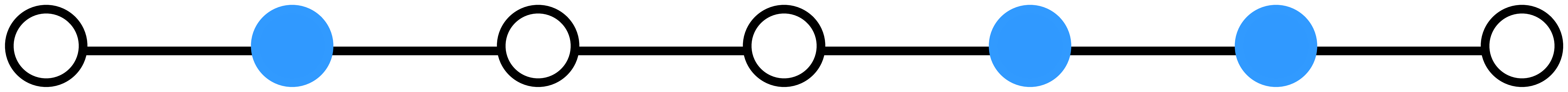

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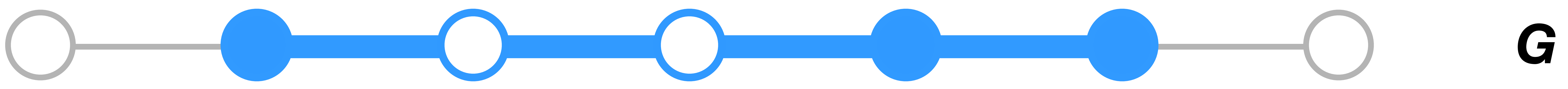


“shortest path convex hull”

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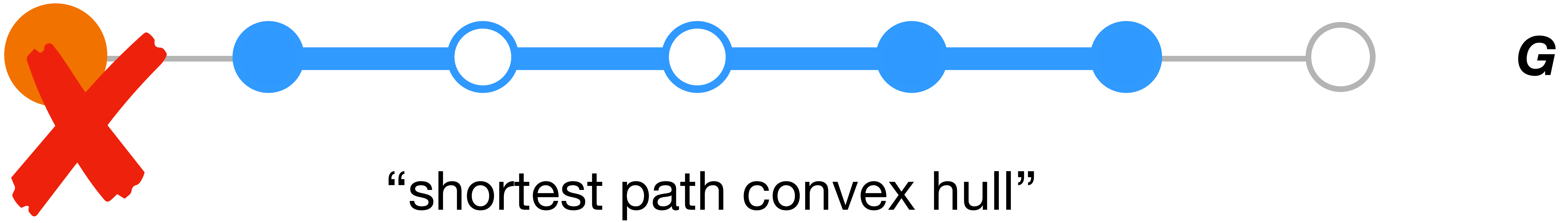


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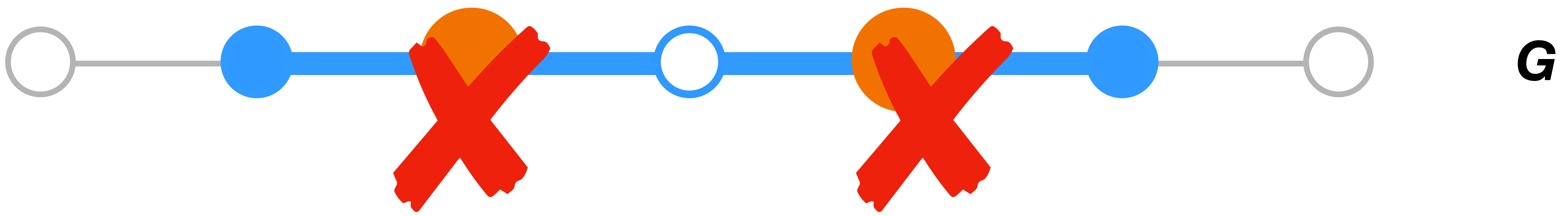

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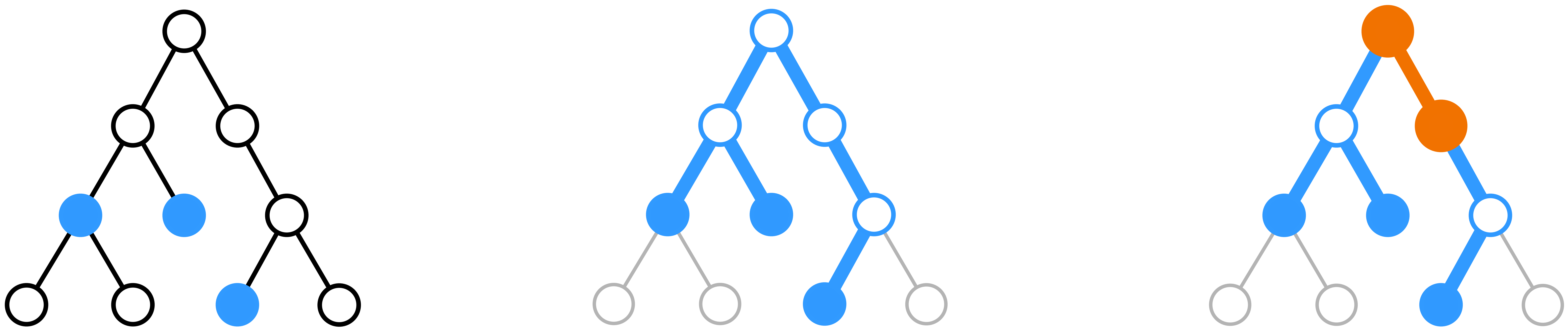

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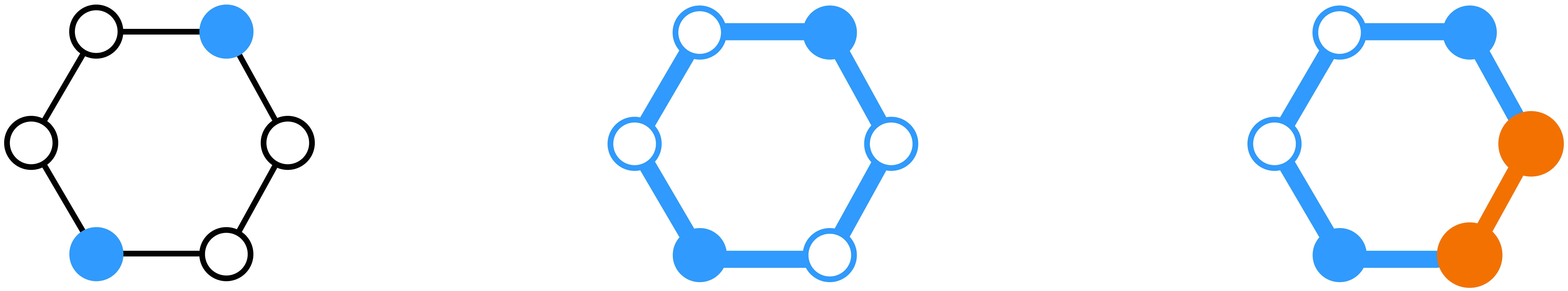

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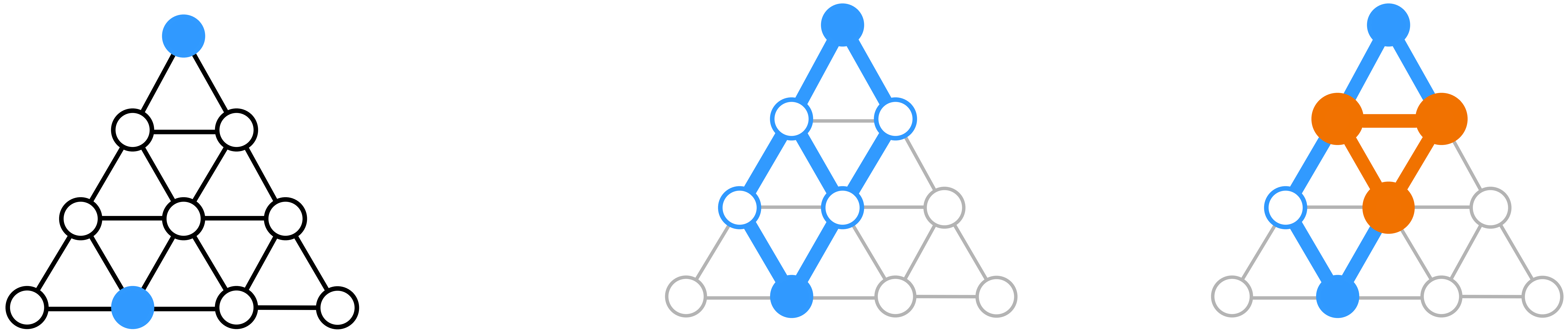

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**Nowak and Rybicki (DISC 2019)**
Message-passing + arbitrary faults

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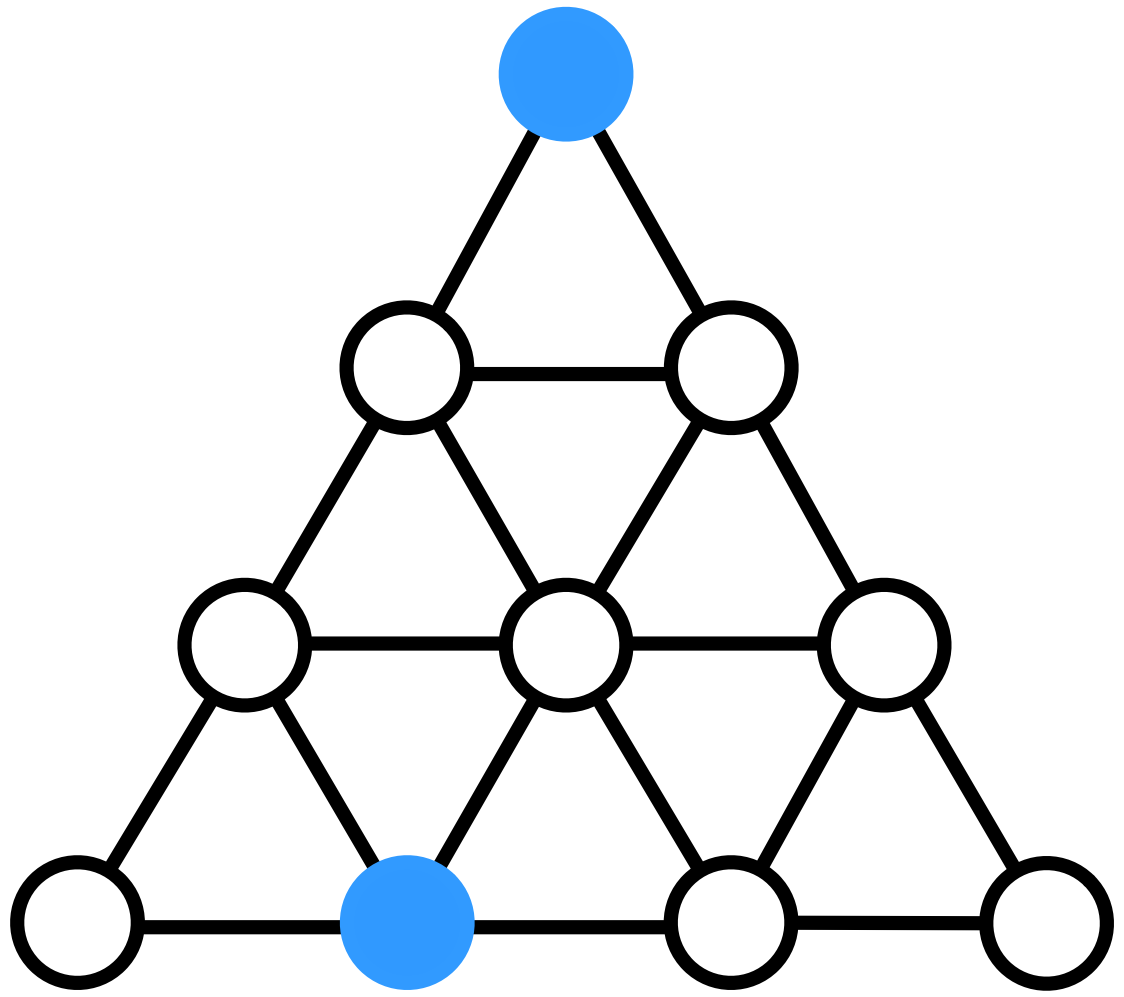

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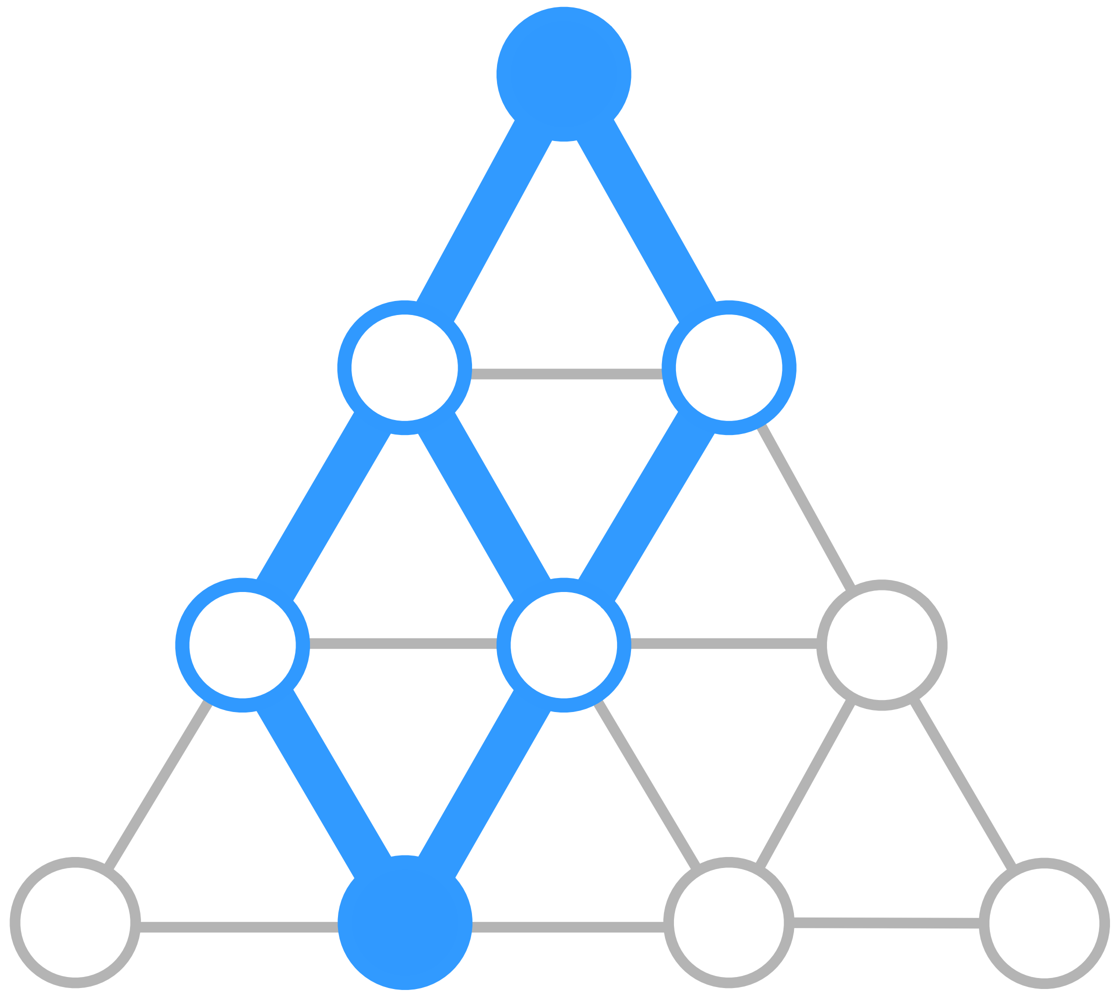

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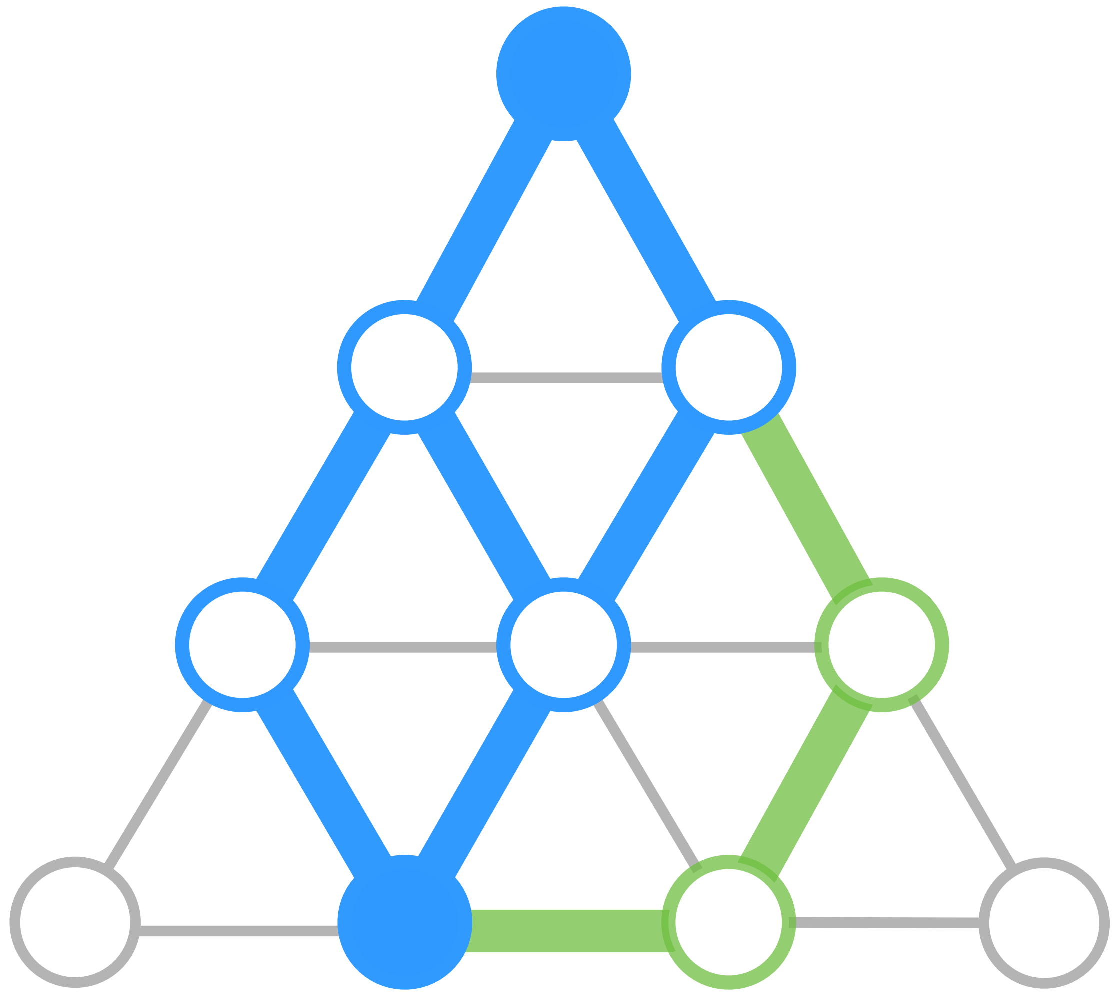

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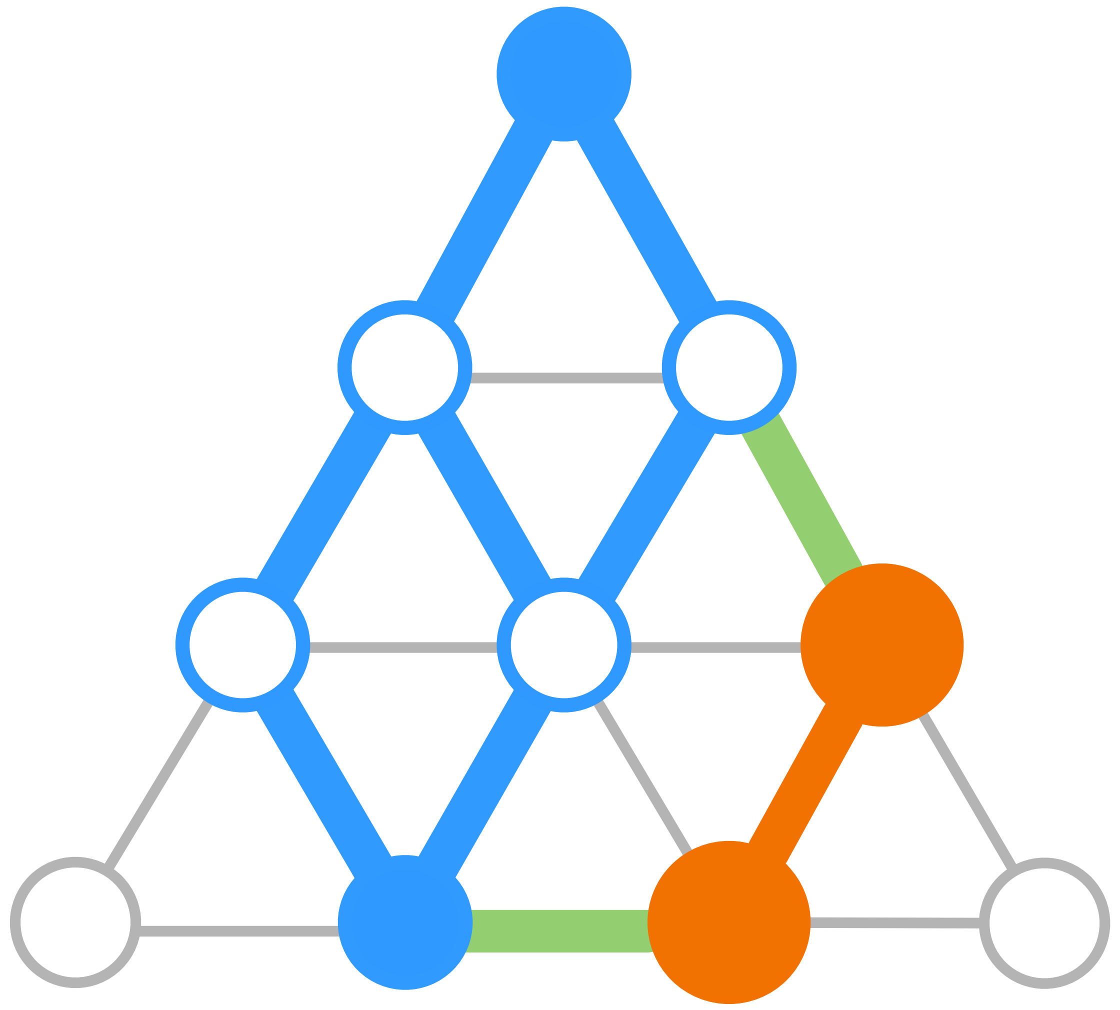

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# **Other validity conditions:**
clique validity

**Alcántara, Castañeda, Flores-Peñaloza, and Rajsbaum (Distributed Computing 2019)**
robots in look-compute-move models

# Other validity conditions:
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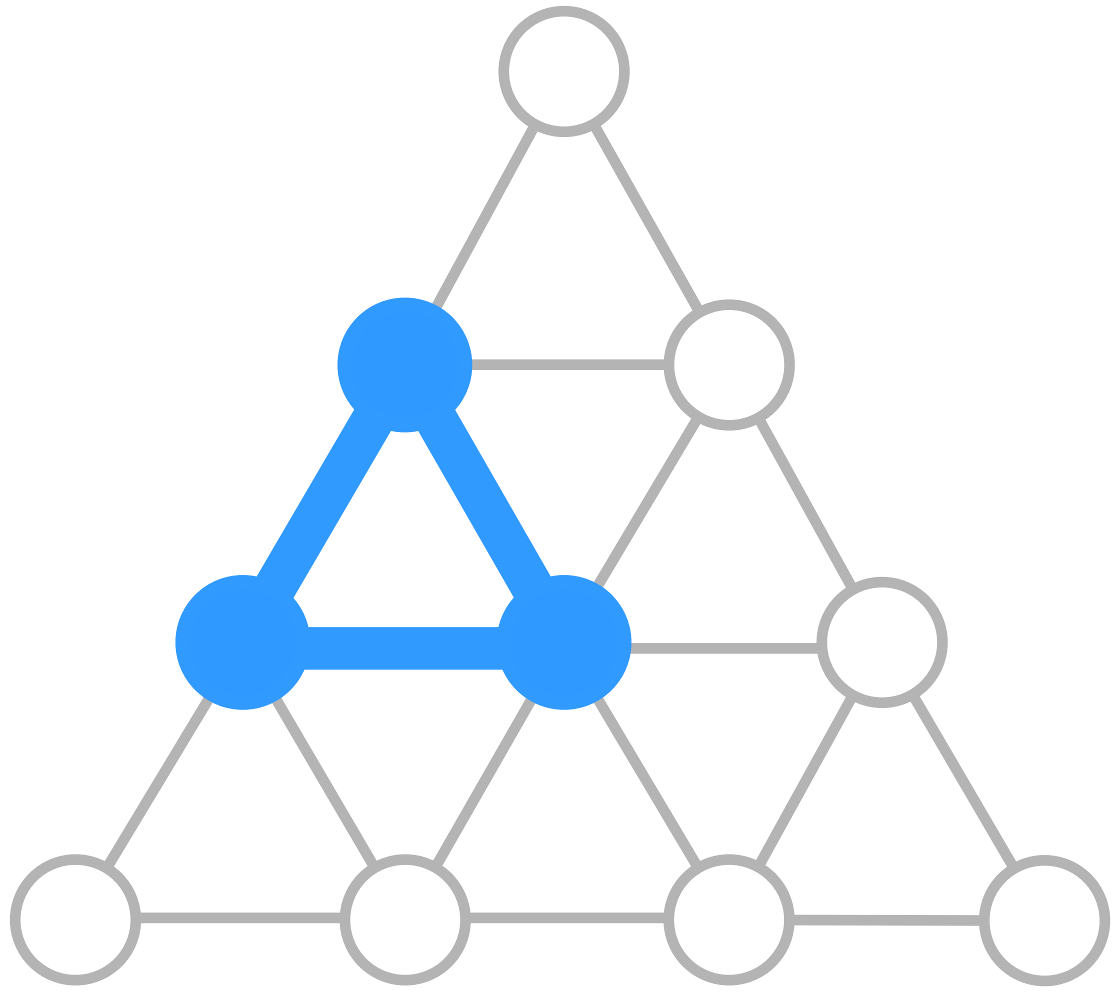

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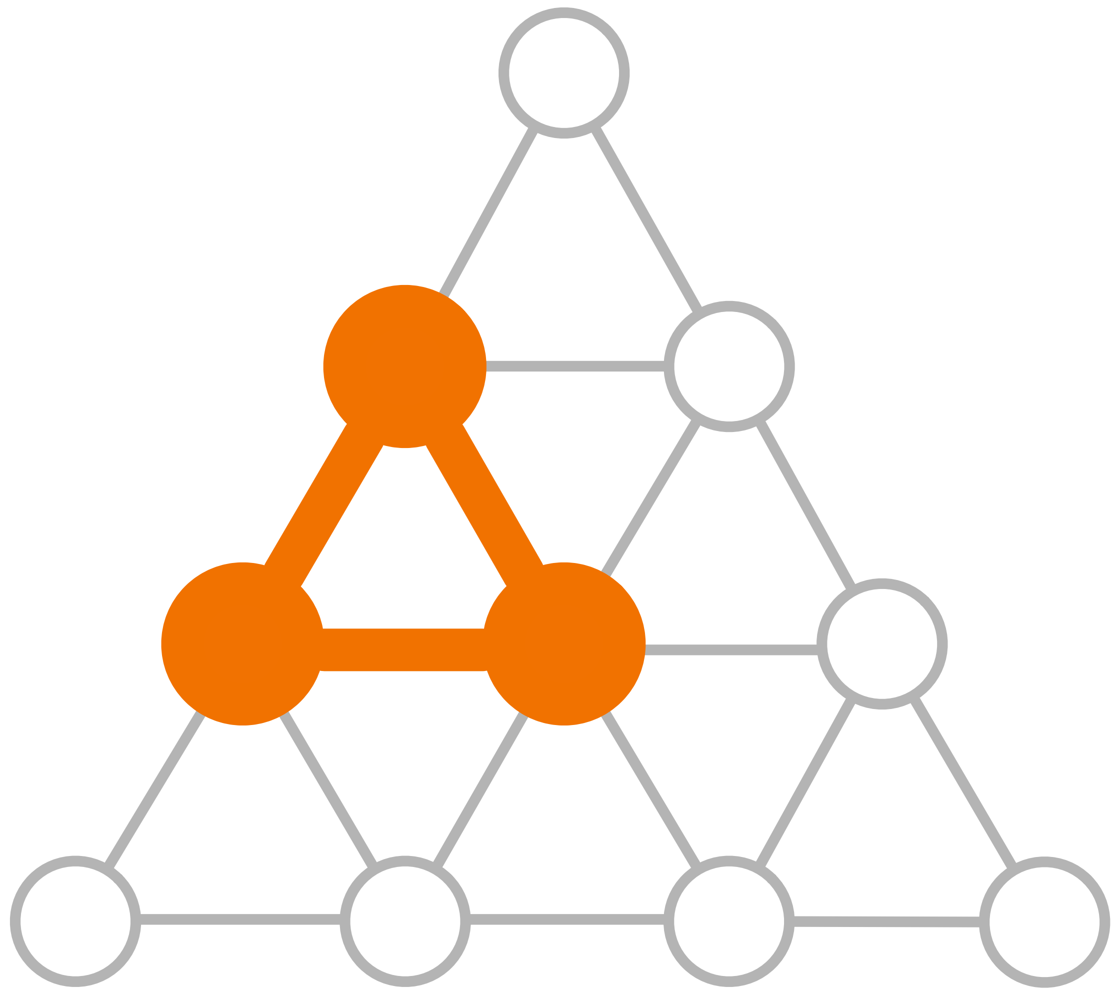

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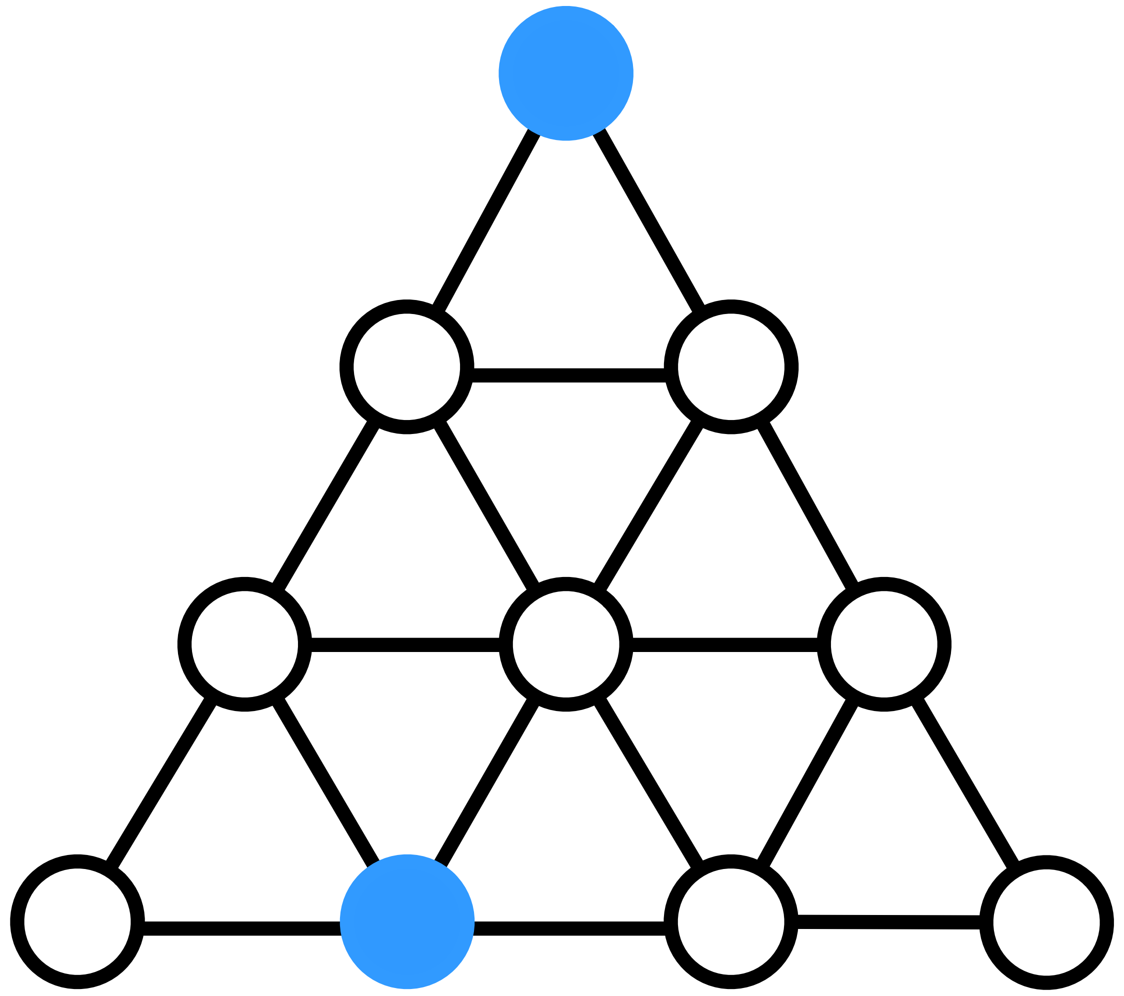

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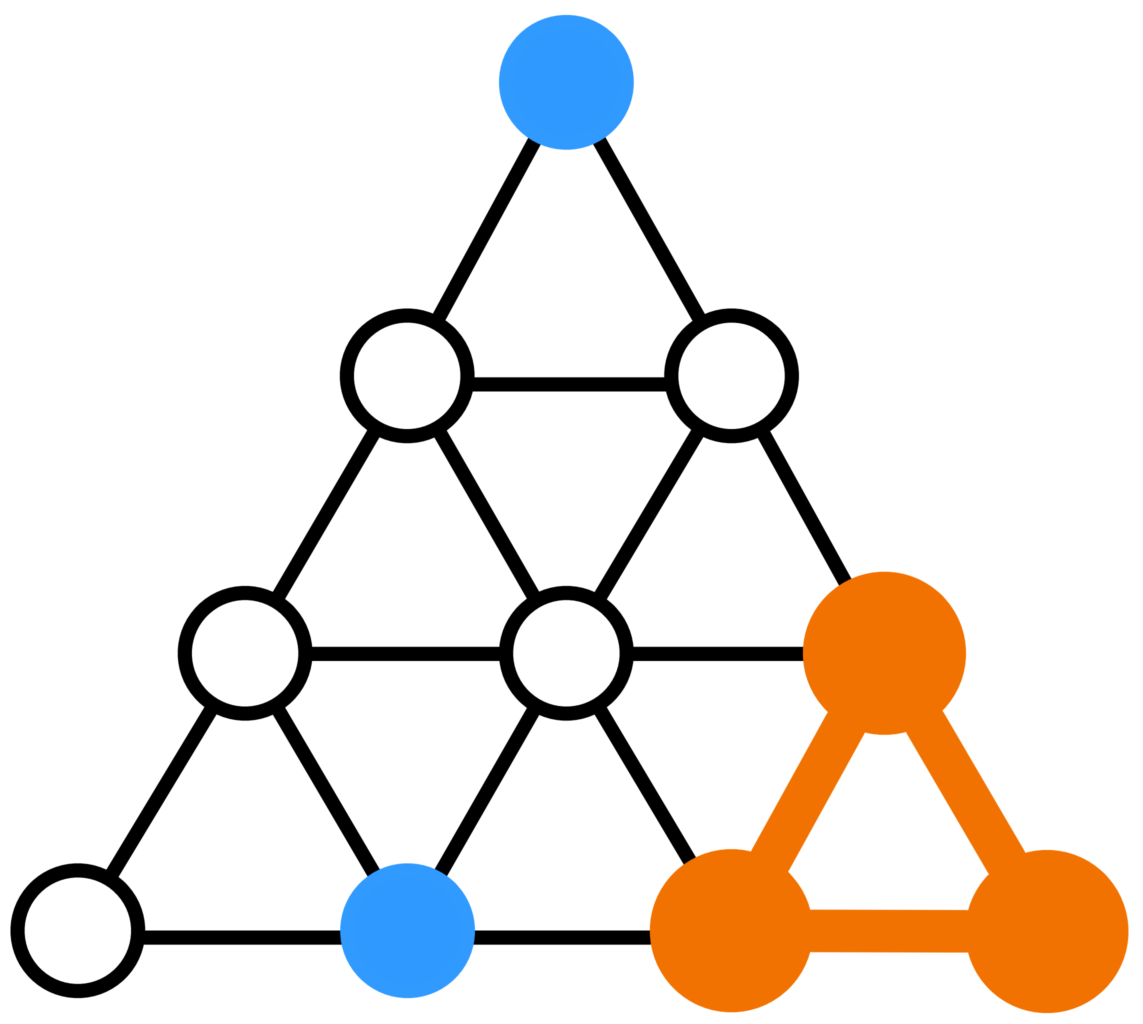

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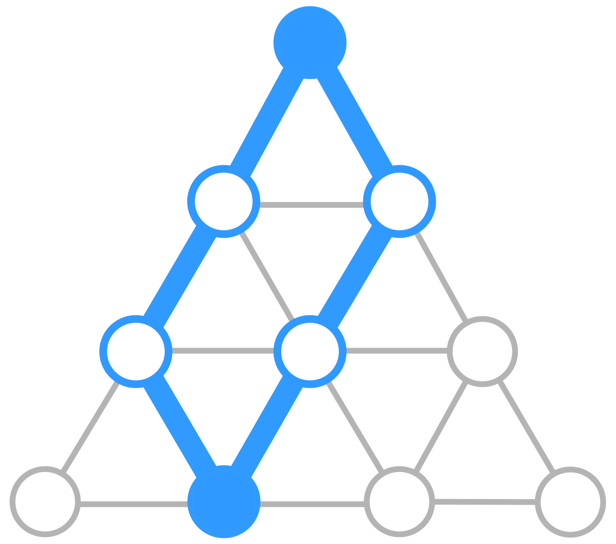
# Comparison of validity conditions

**Shortest path validity:**
each **output** on a *shortest path* between two **inputs**



**Minimal path validity:**
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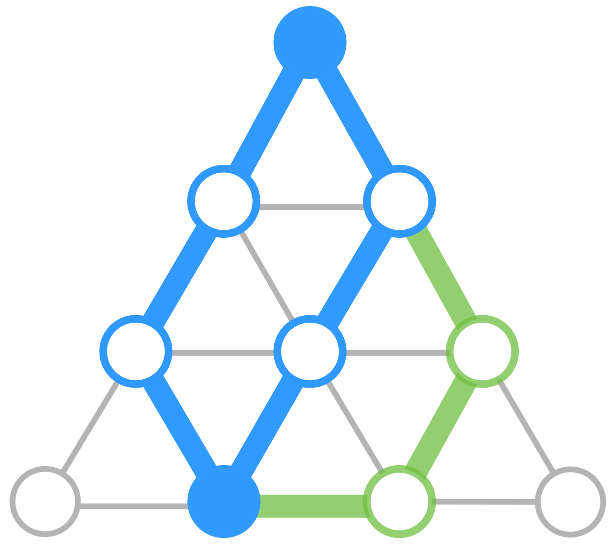
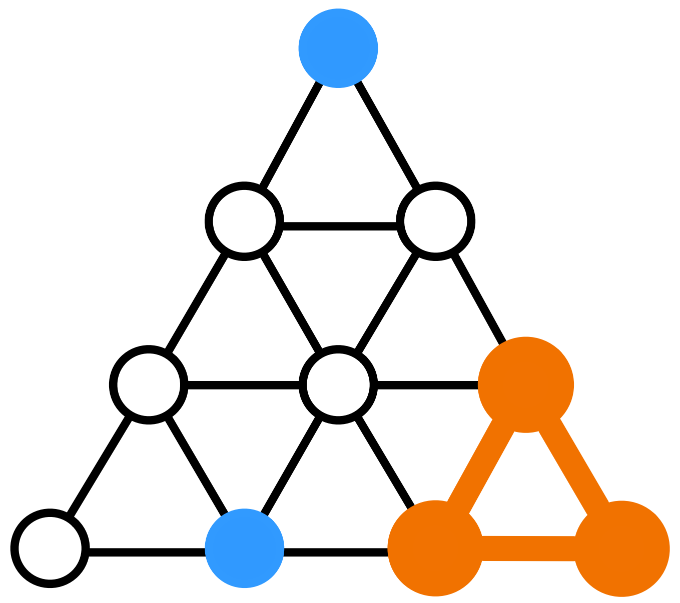


**Clique validity:**
If set $X$ of **inputs** forms a clique in $G$, then $Y \subseteq X$



# Comparison of validity conditions

**Shortest path validity:**
each **output** on a *shortest path* between two **inputs**

**Nice for upper bounds!**

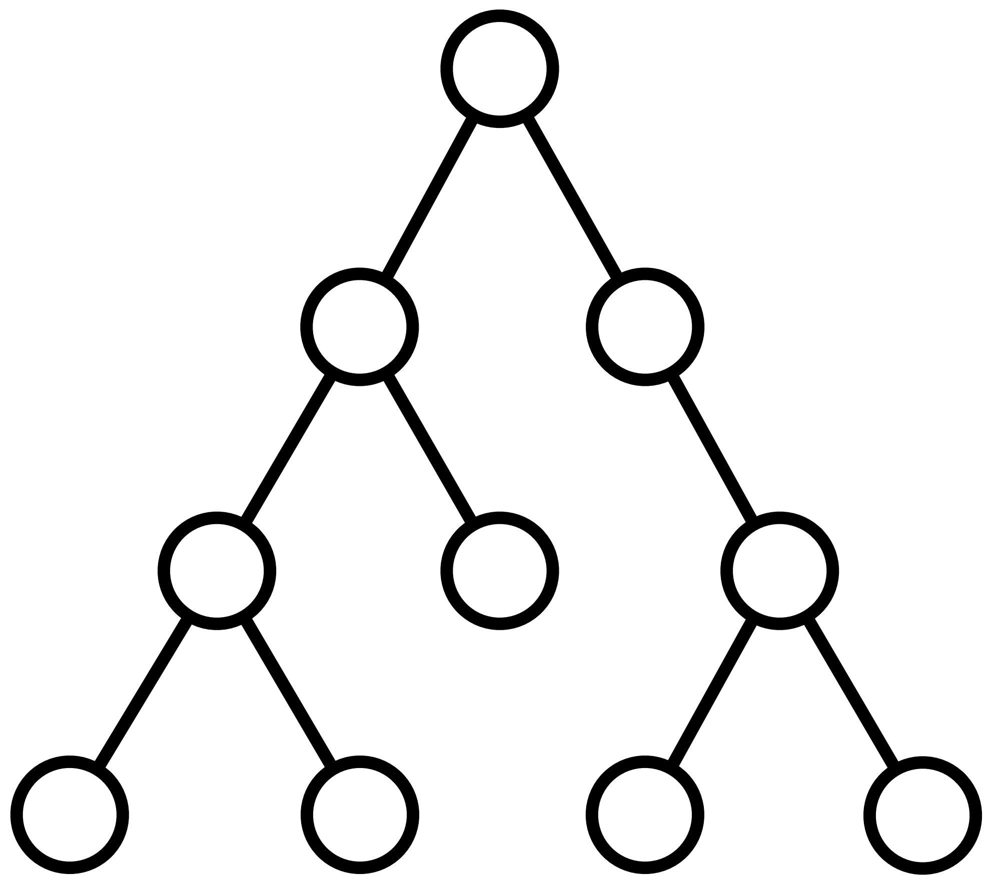
**Minimal path validity:**
Each **output** on a *minimal path* between **inputs**

**Clique validity:**
If set ***X*** of **inputs** forms a clique in ***G***, then $Y \subseteq X$

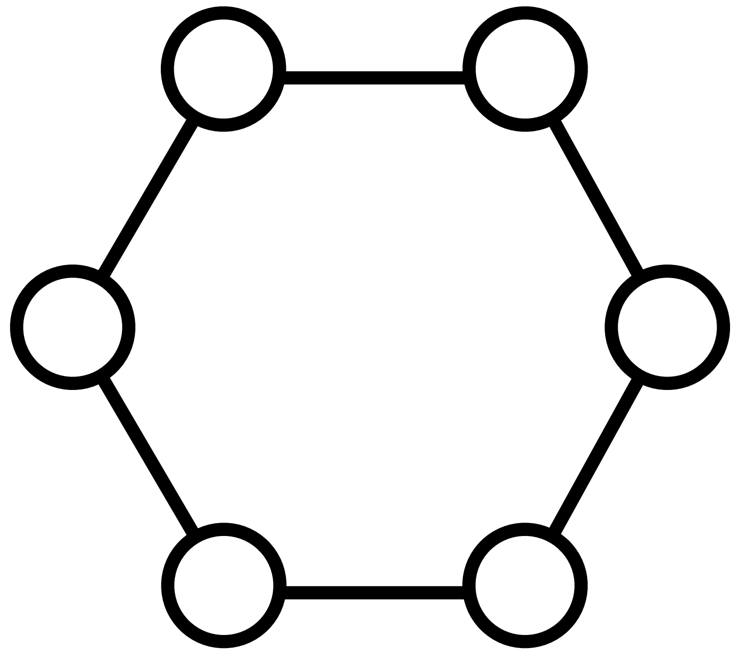
**Nice for lower bounds!**

Some other validity conditions also exist:
see e.g., Alcántara et al. (2019)

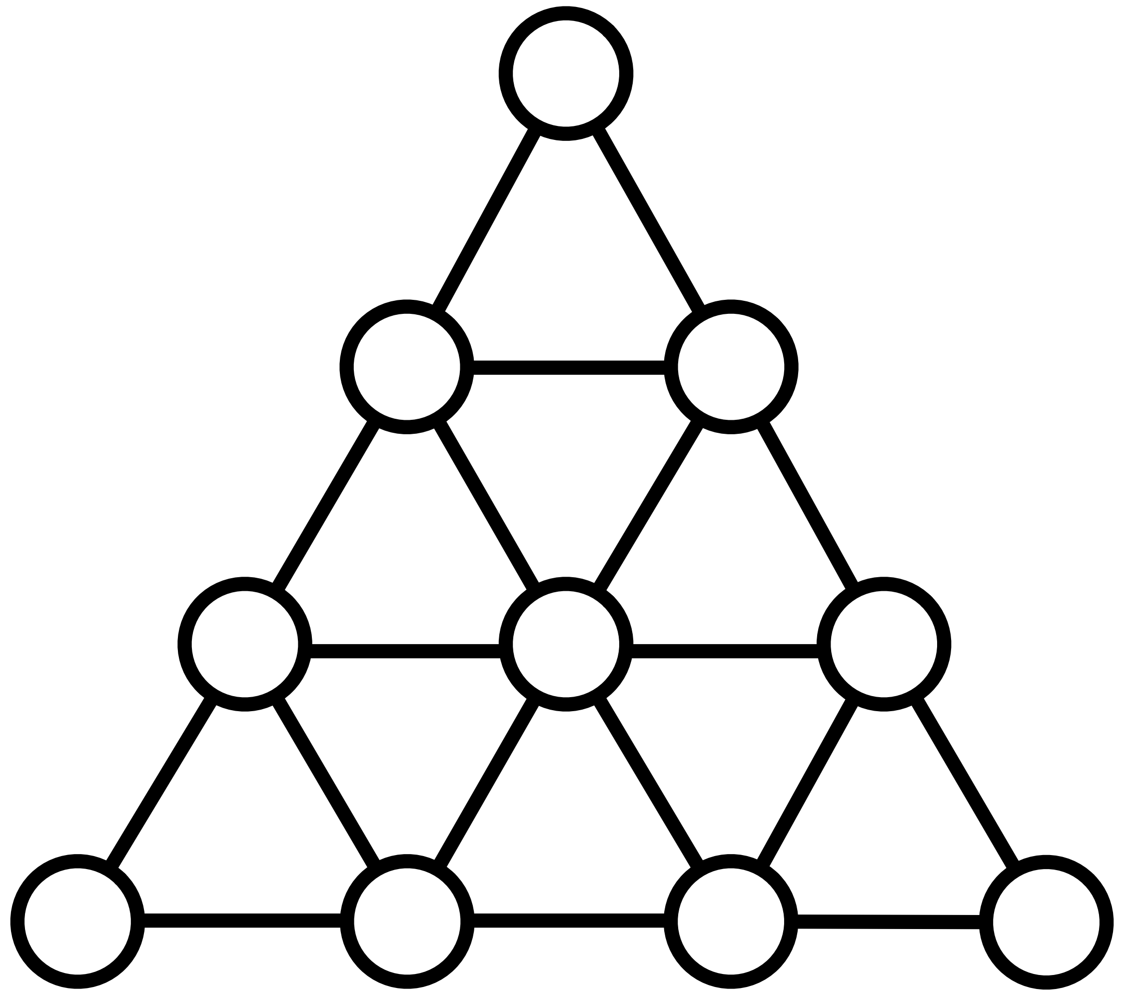
# On what graphs is approximate agreement wait-free solvable?



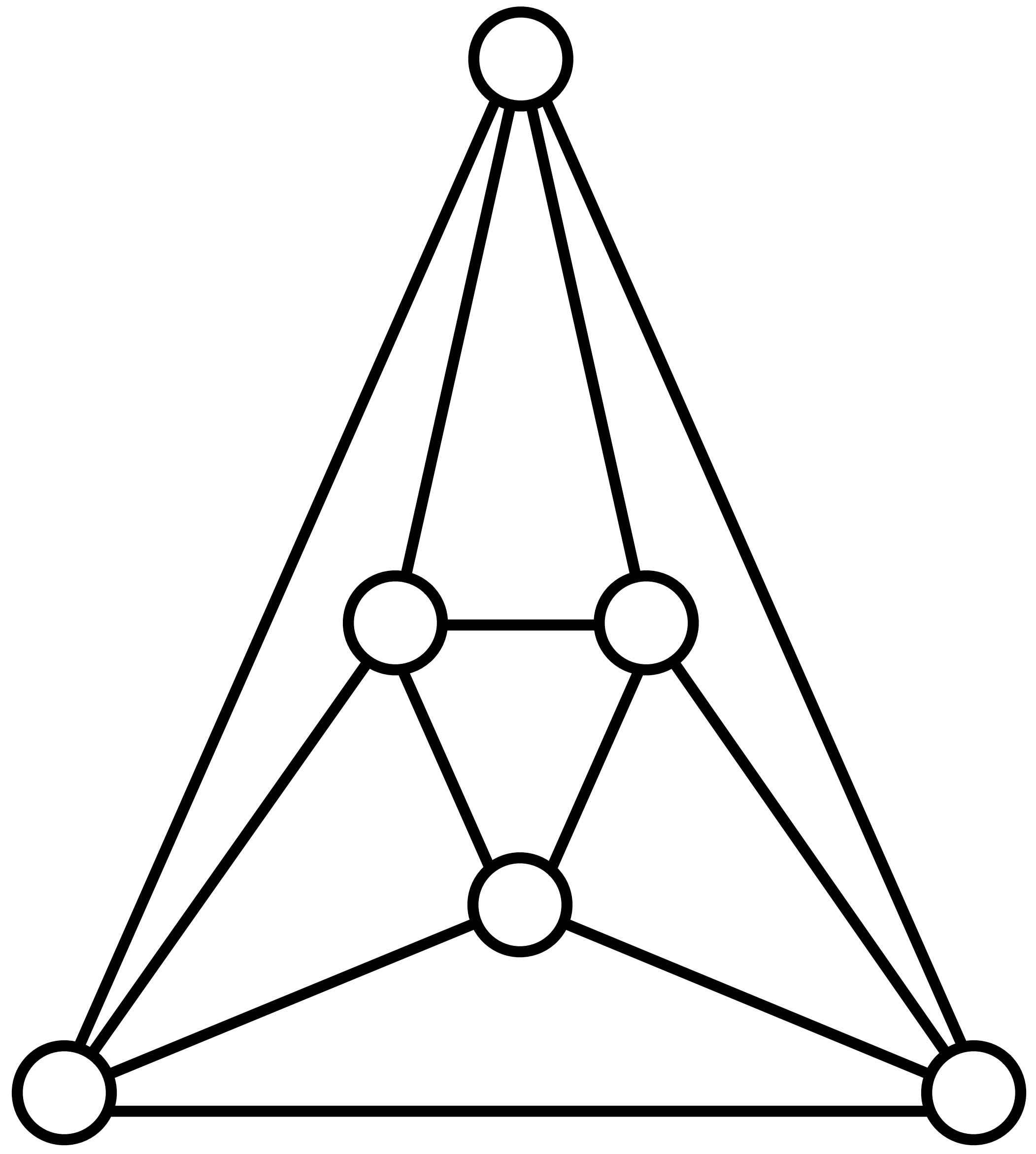
trees



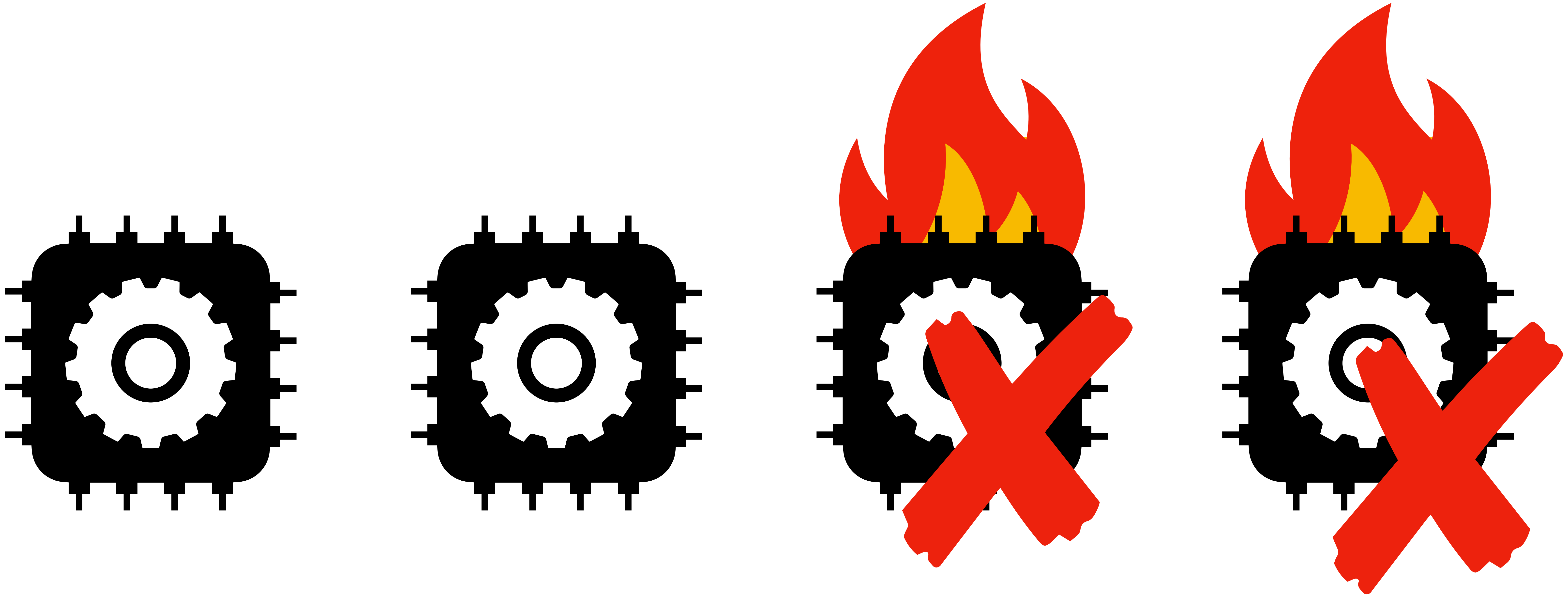
cycles



bridged graphs



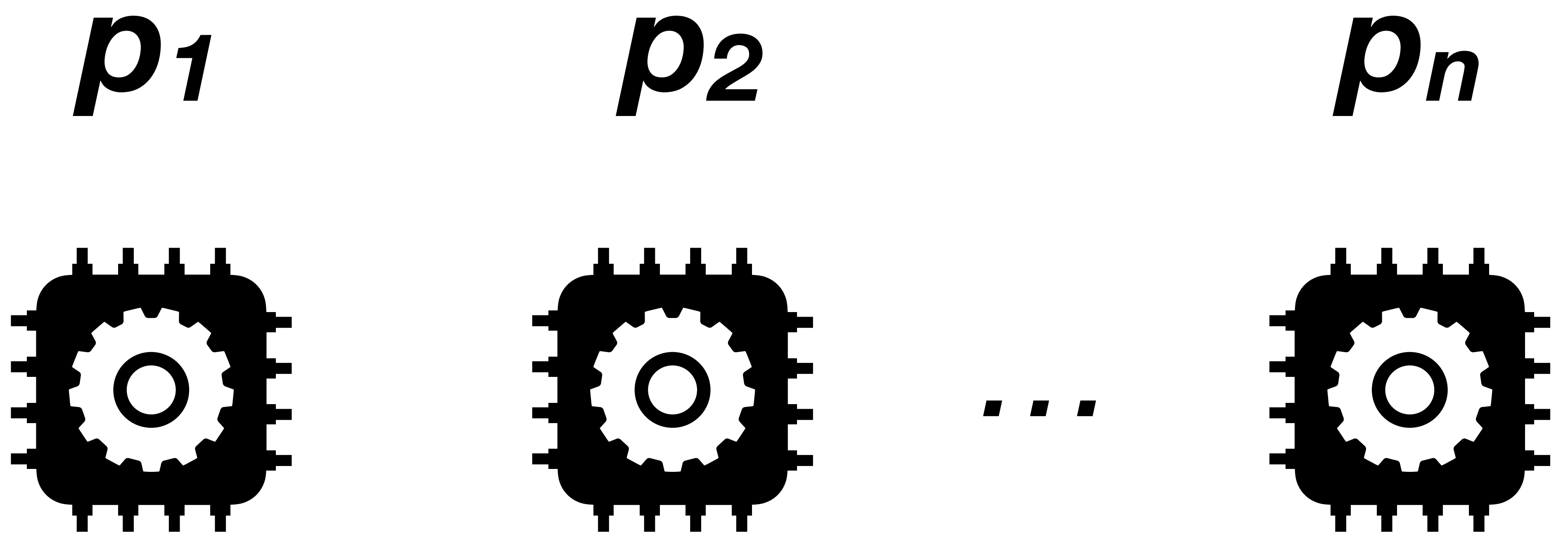
triangulated spheres



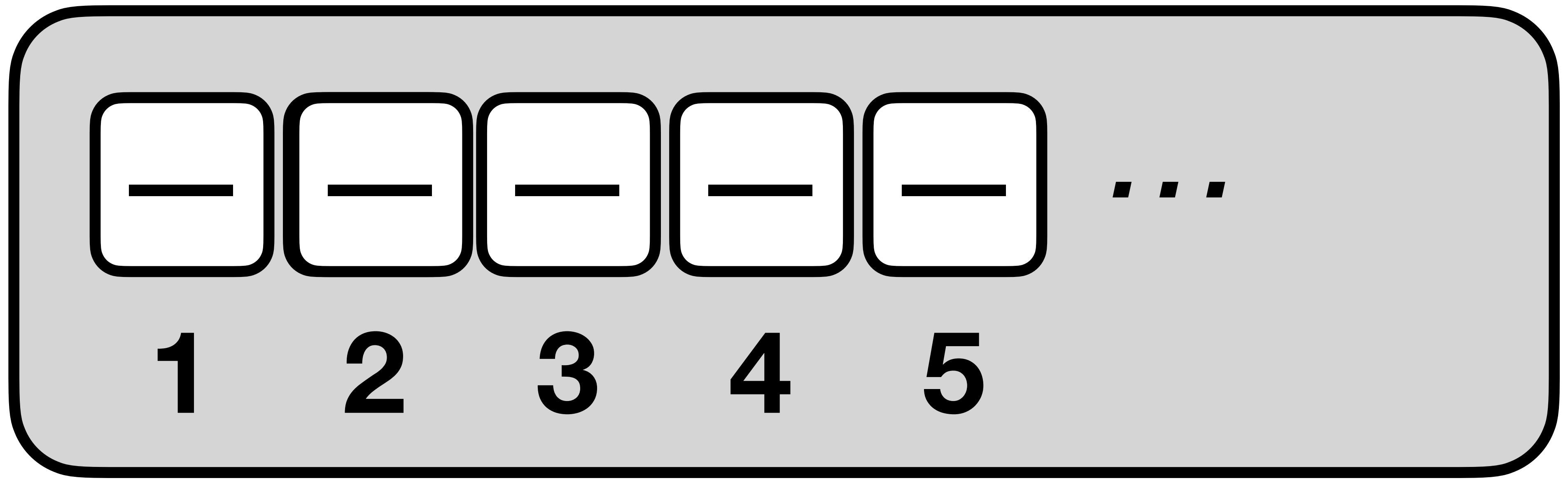
***k*-resilient:** despite at most *k* processes **crashing**,
correct processes terminate with correct outputs

**wait-free:** ($n$-1)-resilient

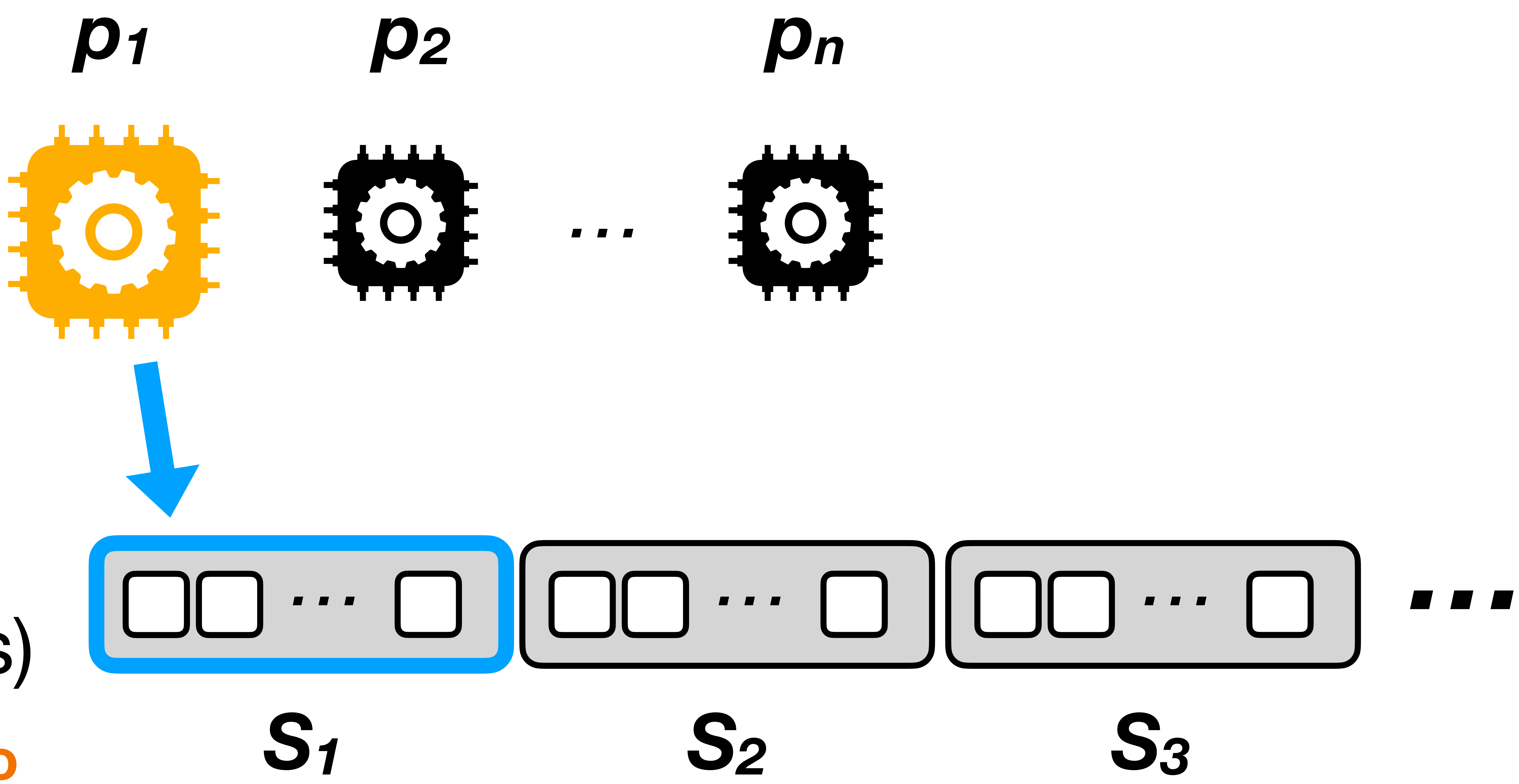
# **The model:**
asynchronous shared memory




**shared memory**
(registers)



# The model:
## iterated snapshot model




**shared memory**
(snapshot objects)

each process **writes to** and **scans** each $S_i$ at most once

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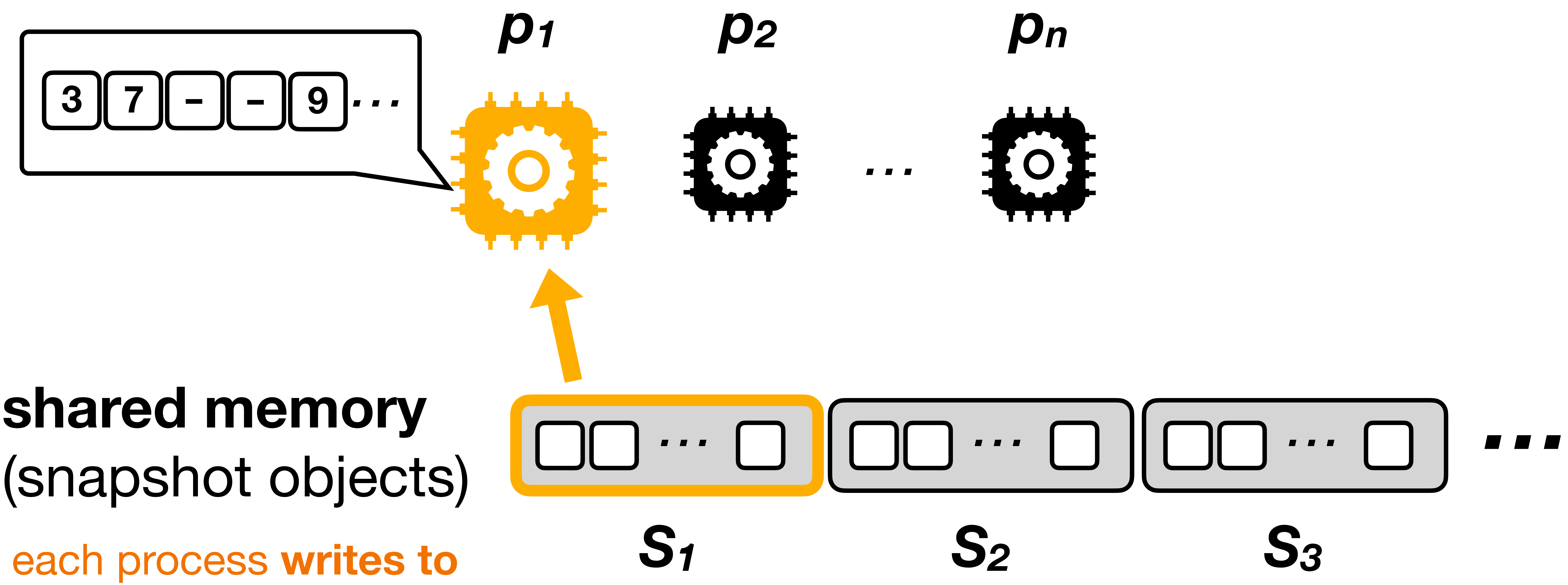


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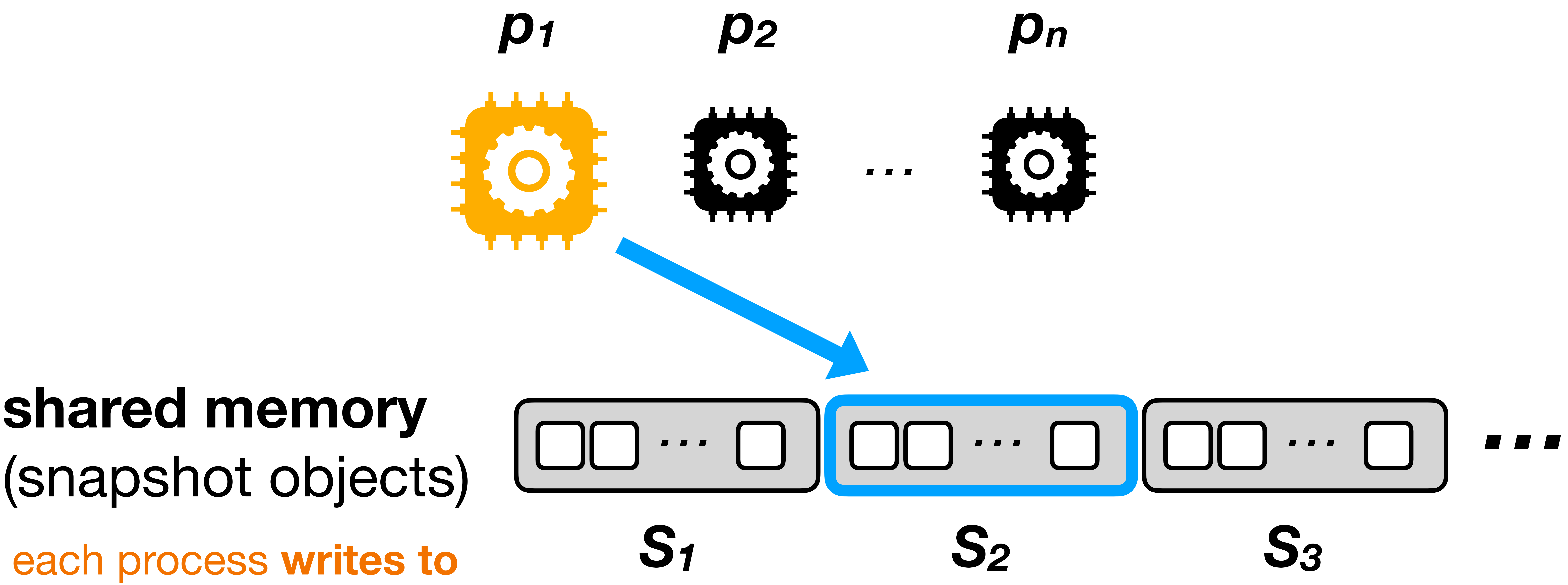


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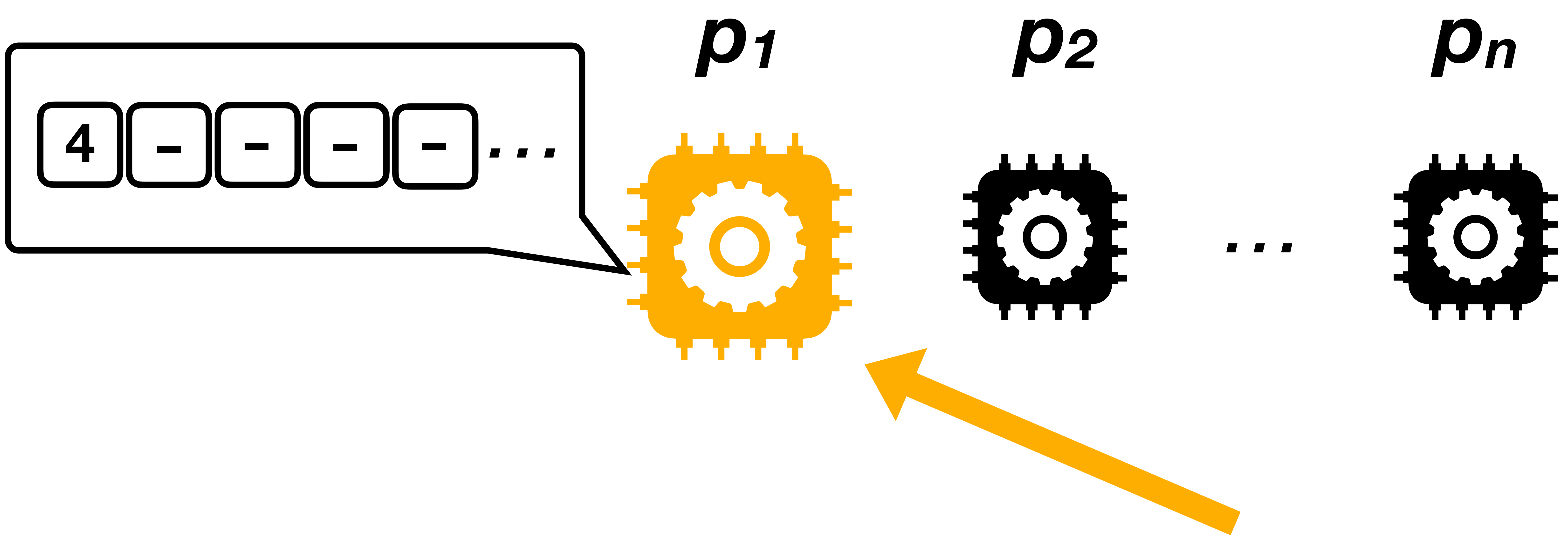
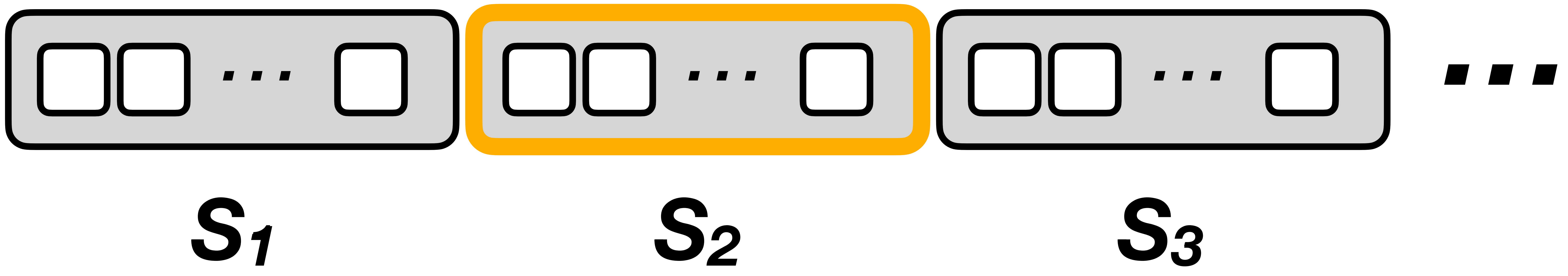
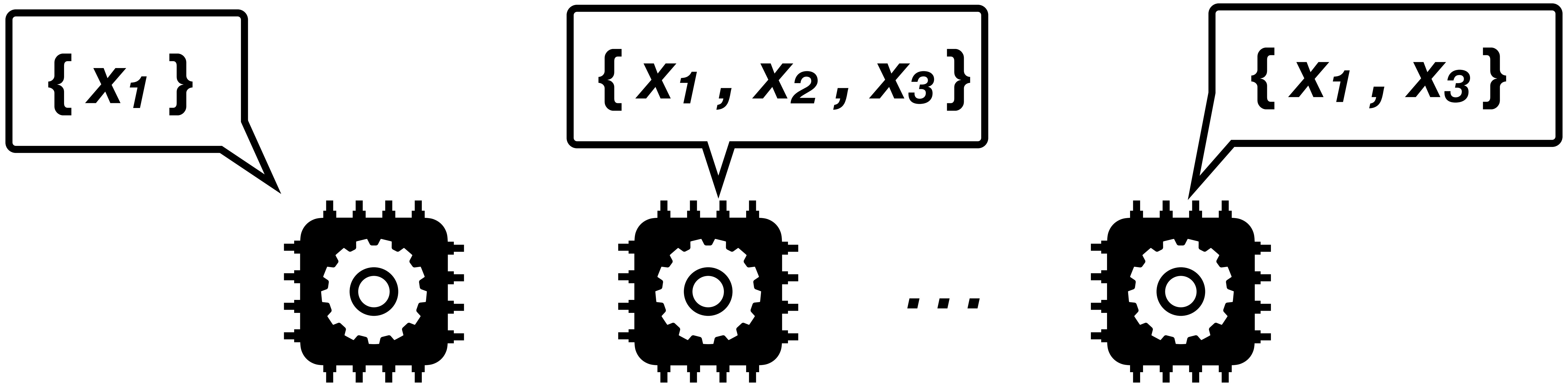


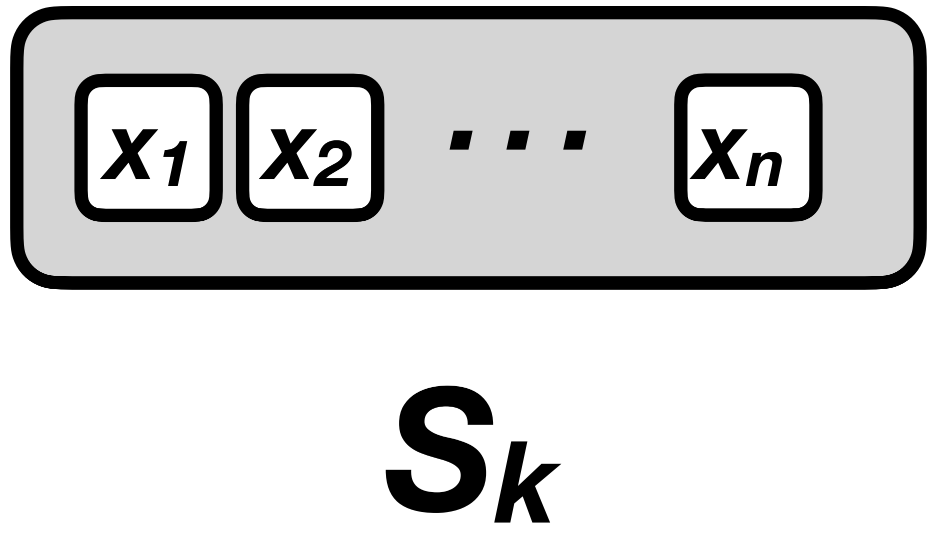

**shared memory**
(snapshot objects)




each process **writes to**
and **scans** each $S_i$ at most once

# **Scanning snapshots:**
# the containment property



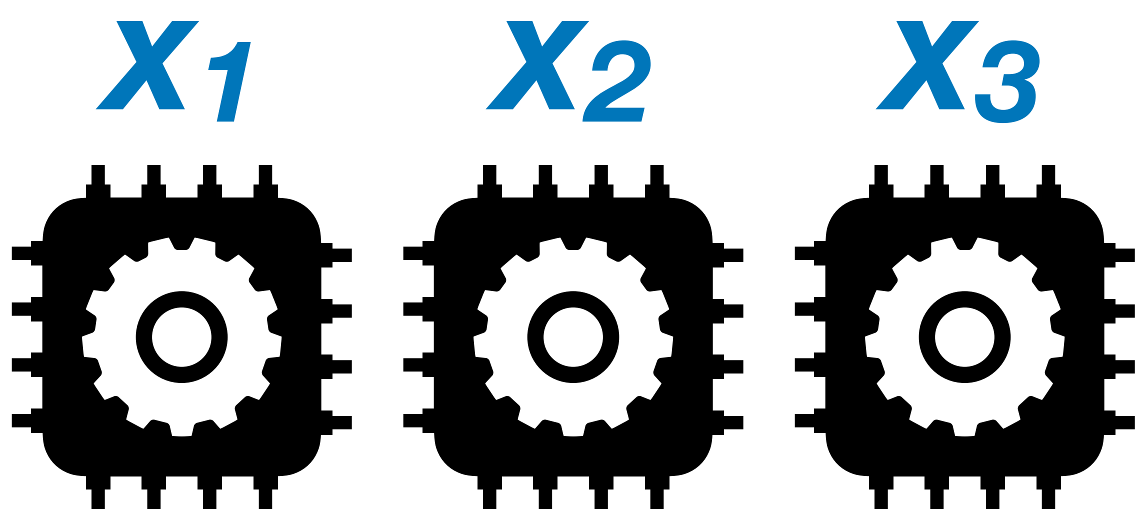



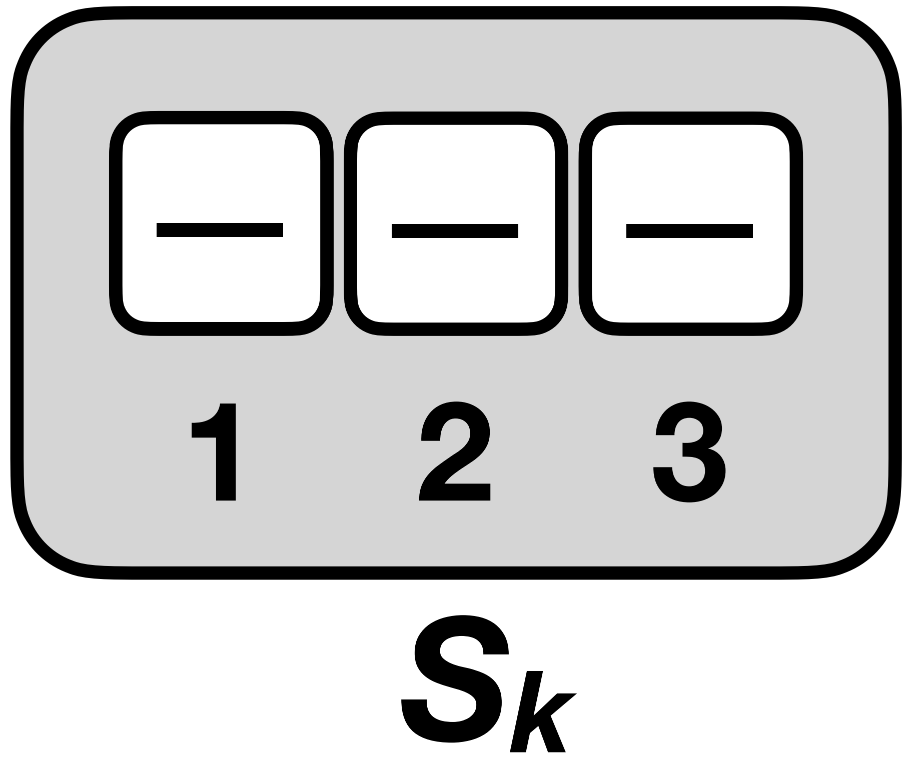

Containment property:

$$\{ x_1 \} \subseteq \{ x_1, x_3 \} \subseteq \{ x_1, x_2, x_3 \}$$

# The general algorithmic approach

# The general shape of algorithms




In iteration $k$ = 1, …, process $p_i$

1. writes $x_i$ to $S_k$
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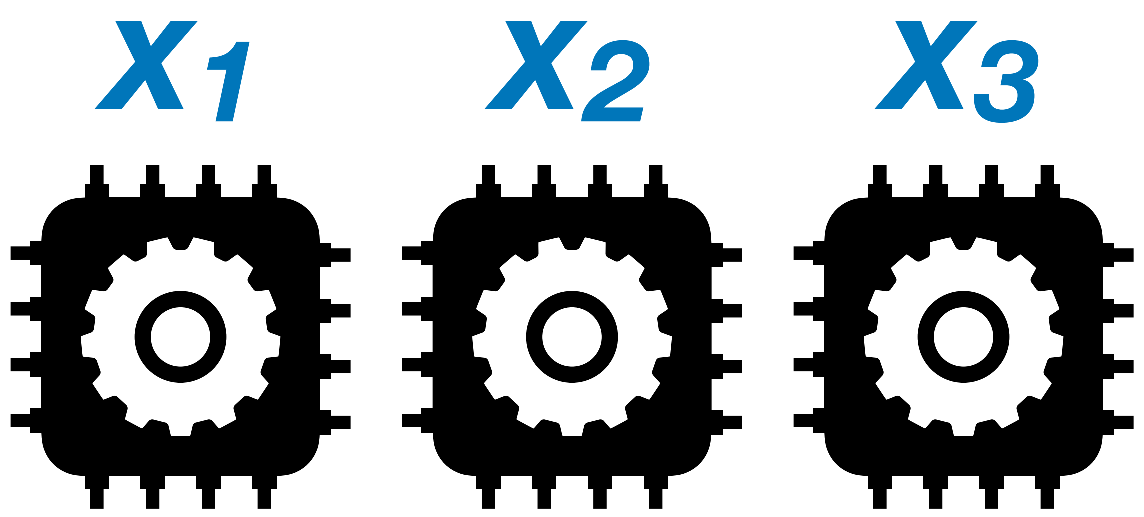
3. sets $x_i$ to $g( V_i )$

for suitable $g : 2^V \to V$

# The general shape of algorithms

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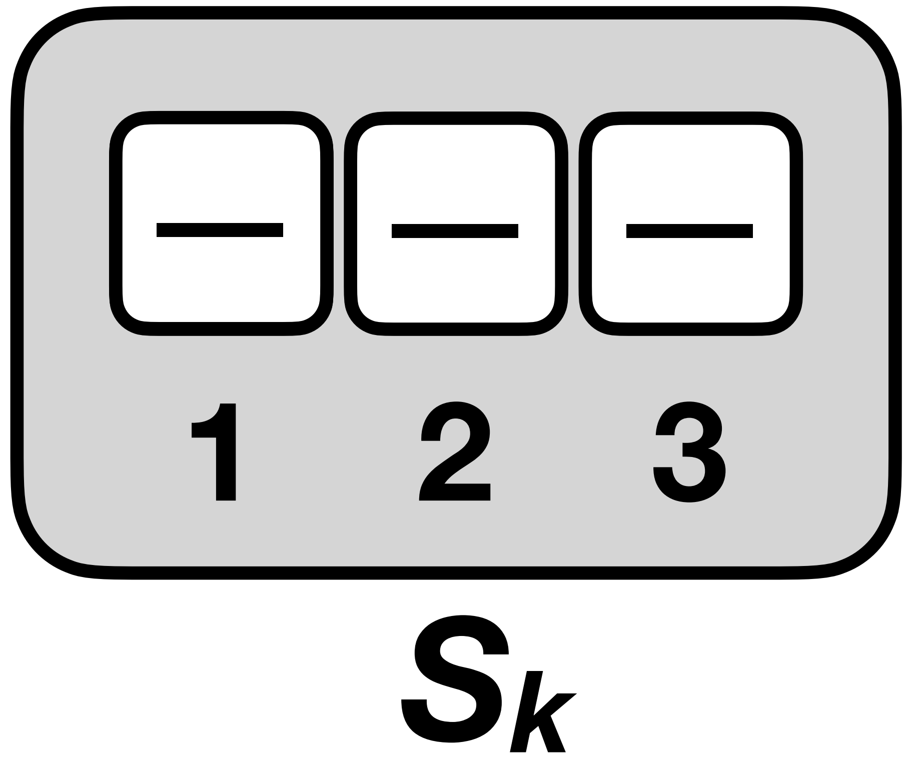
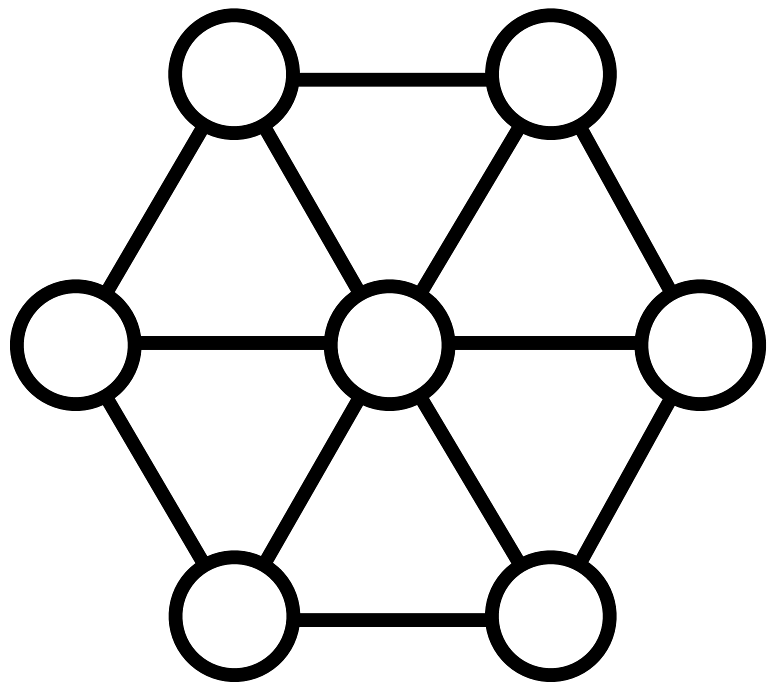
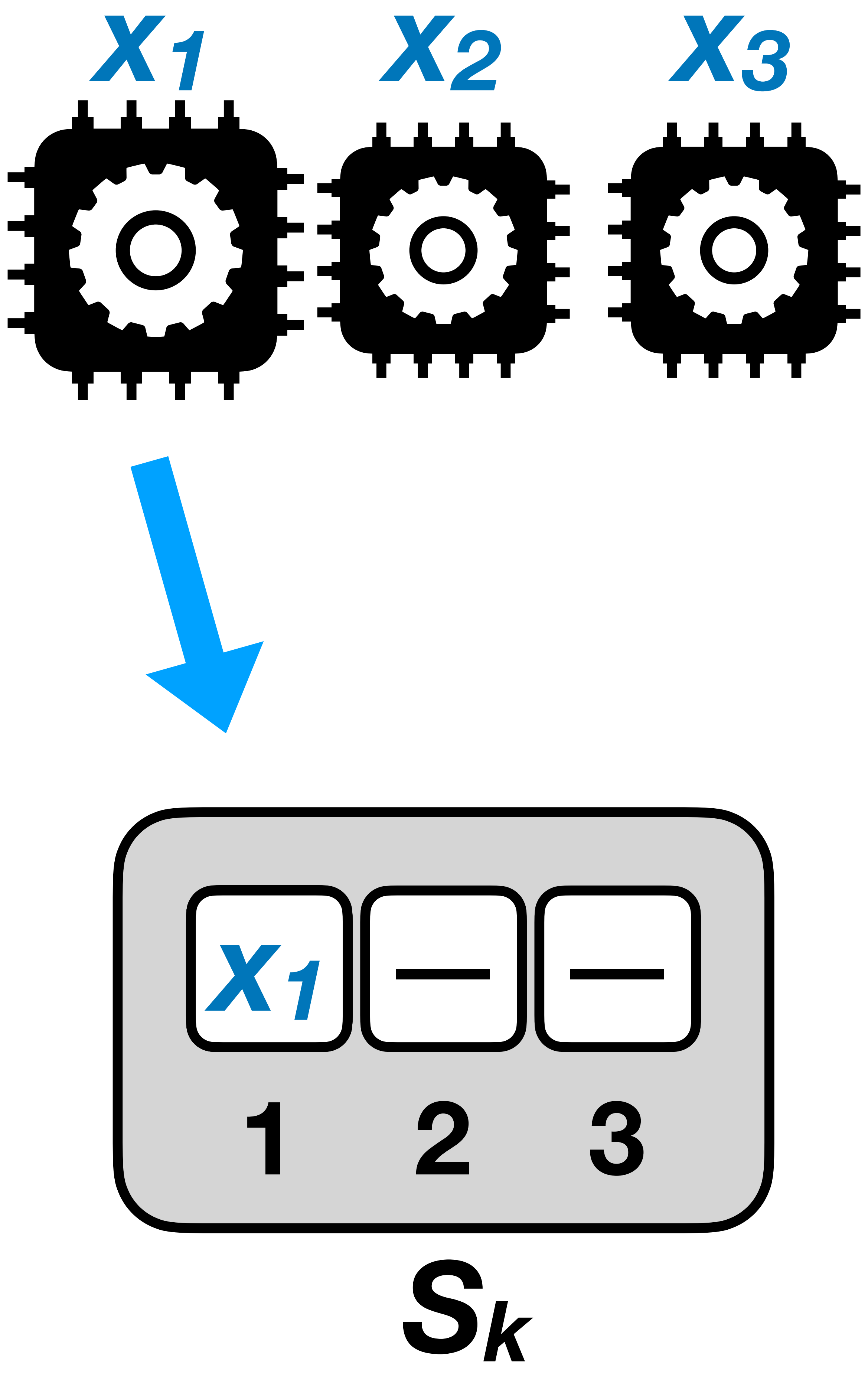
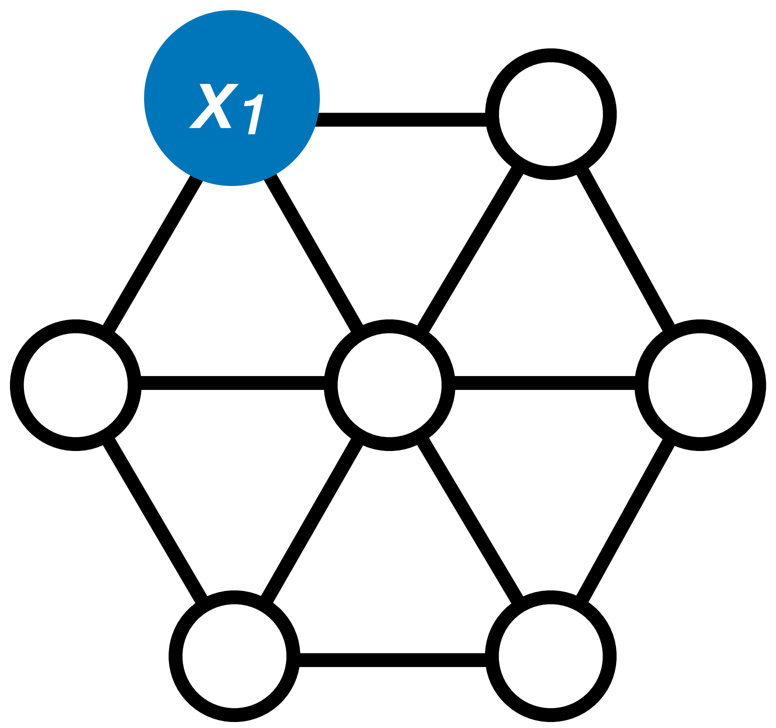
for suitable $\boldsymbol{g : 2^V \to V}$



# The general shape of algorithms

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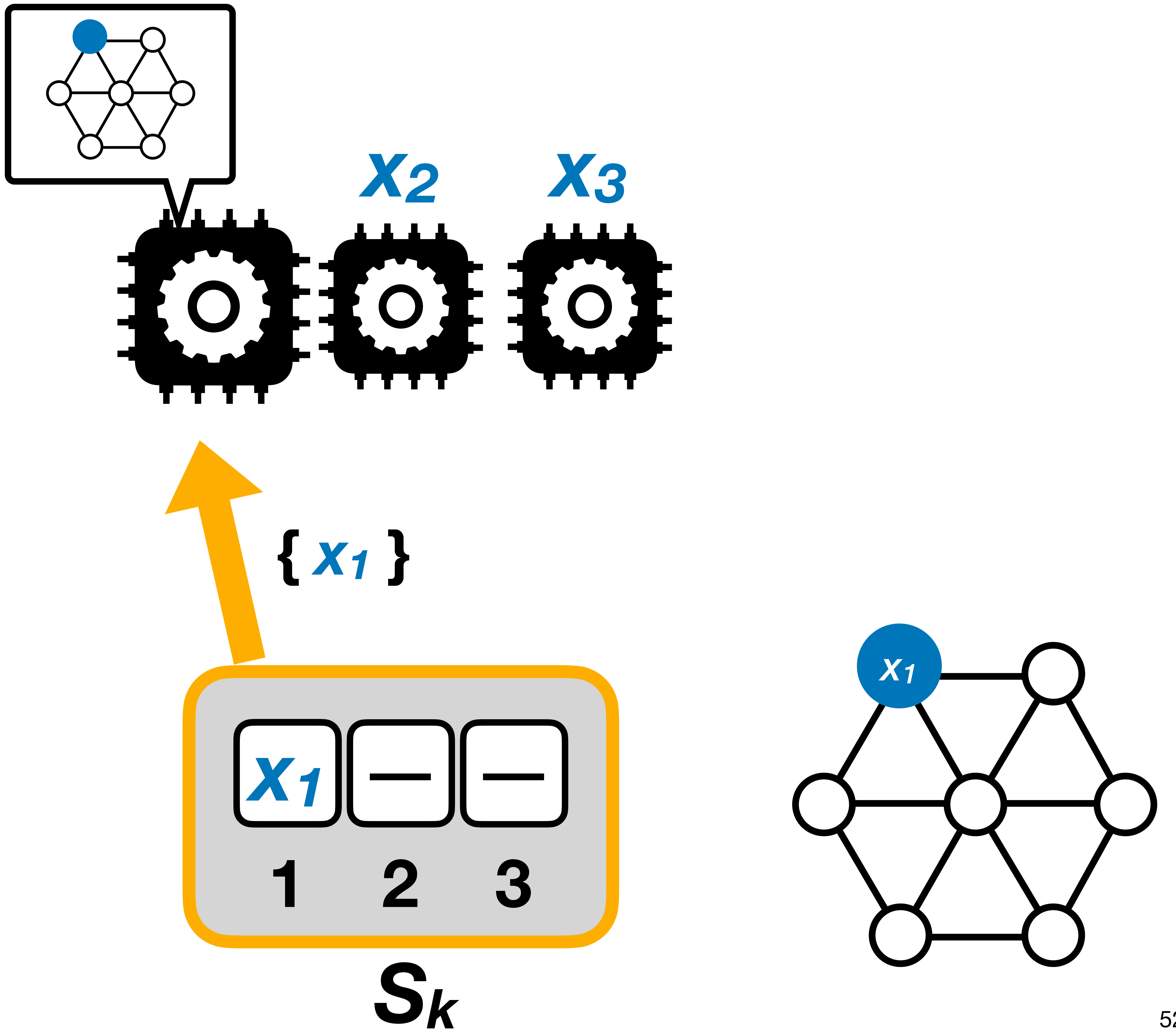


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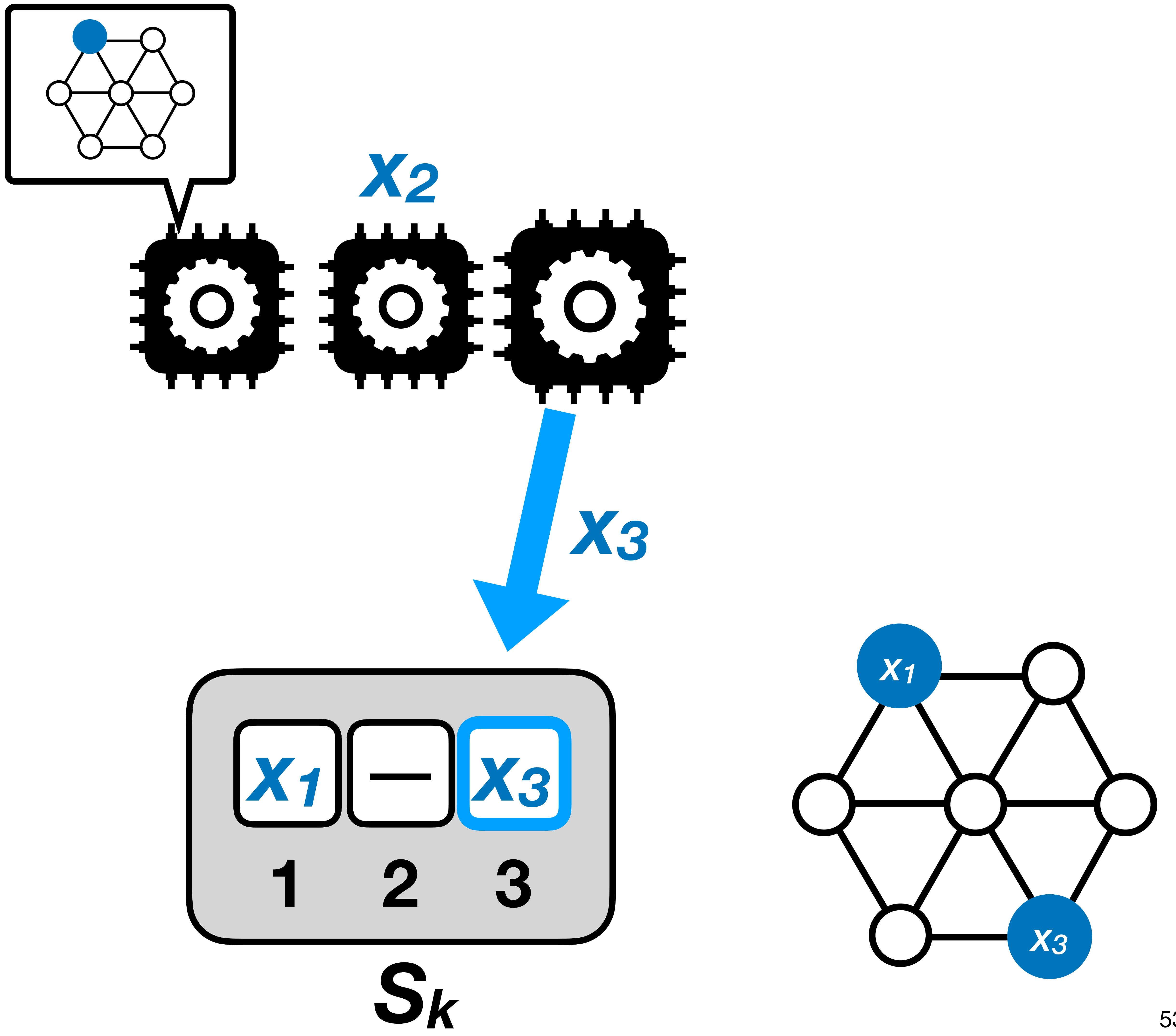


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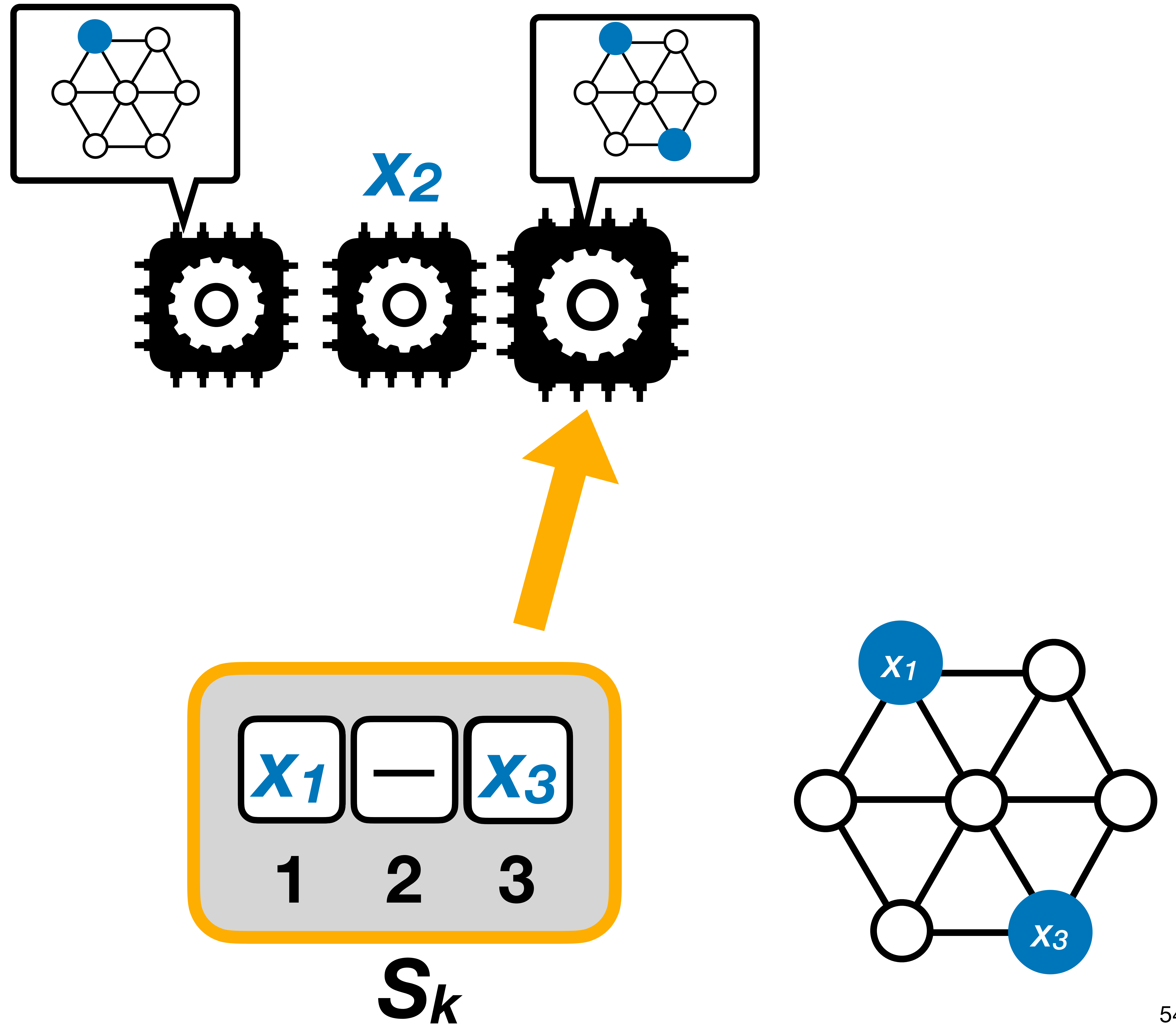


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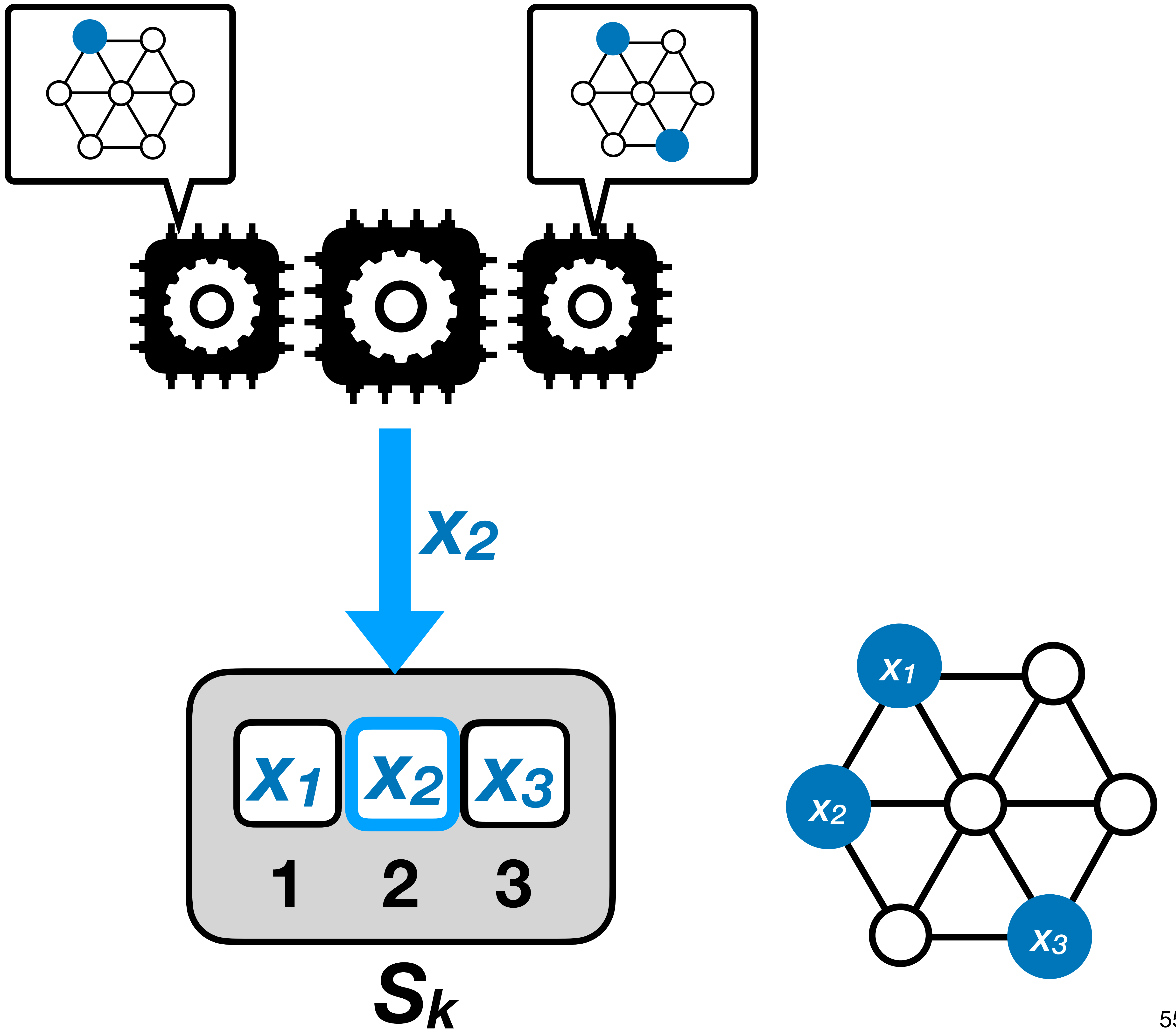


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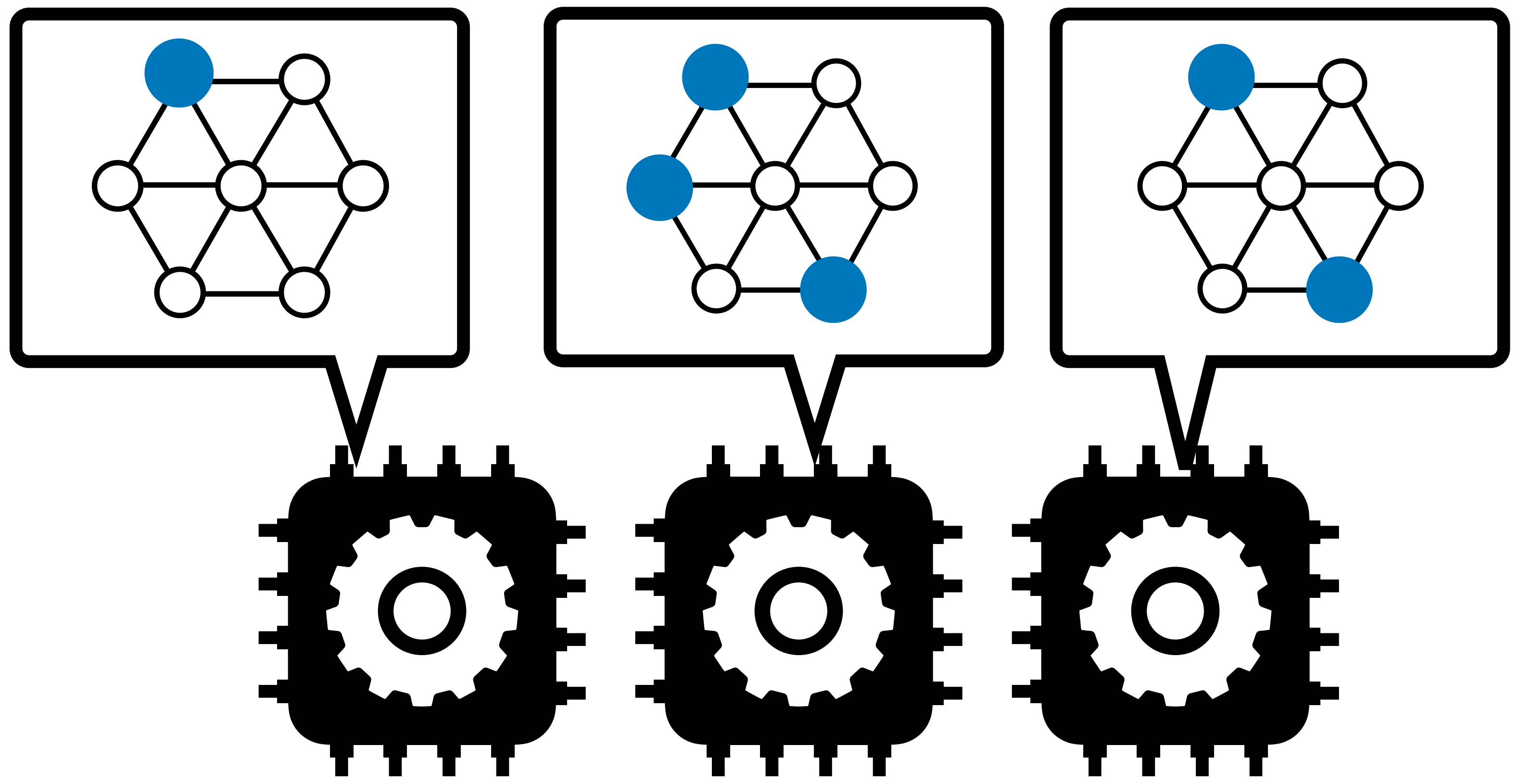

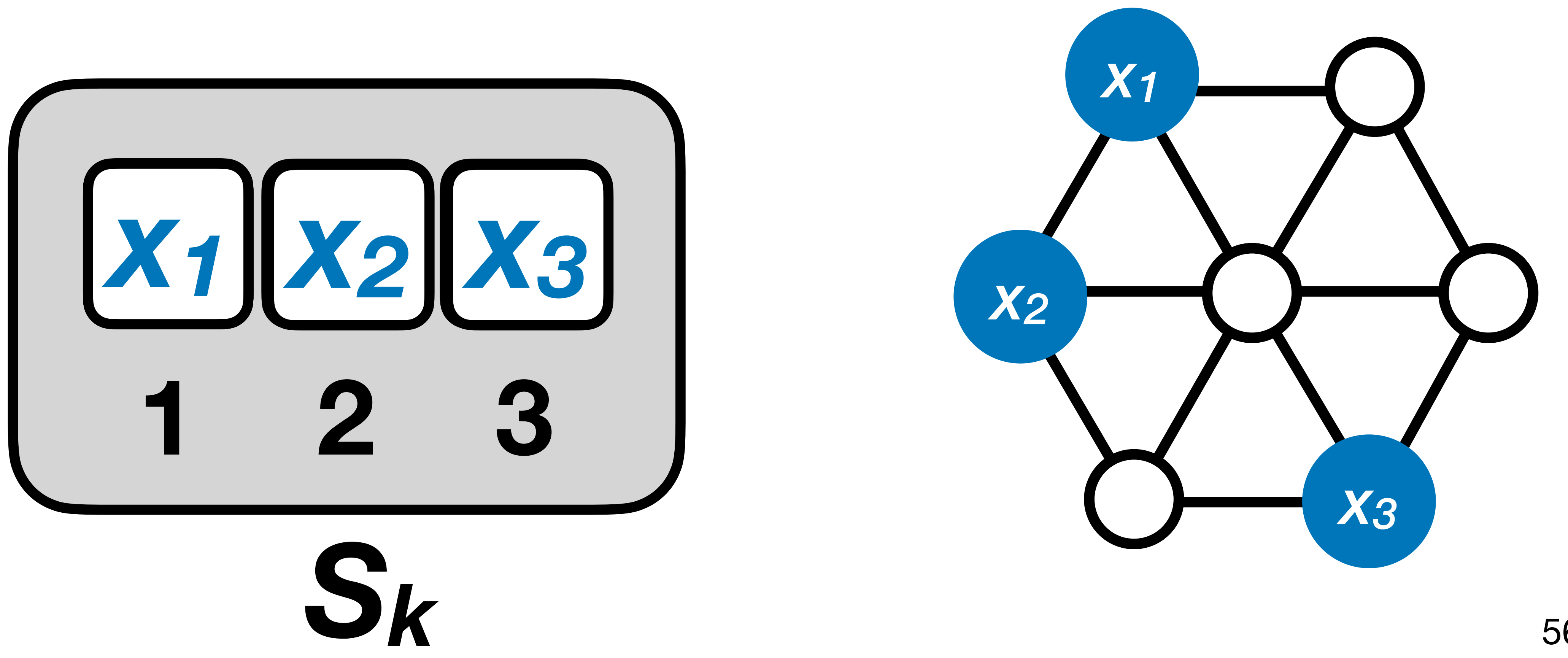


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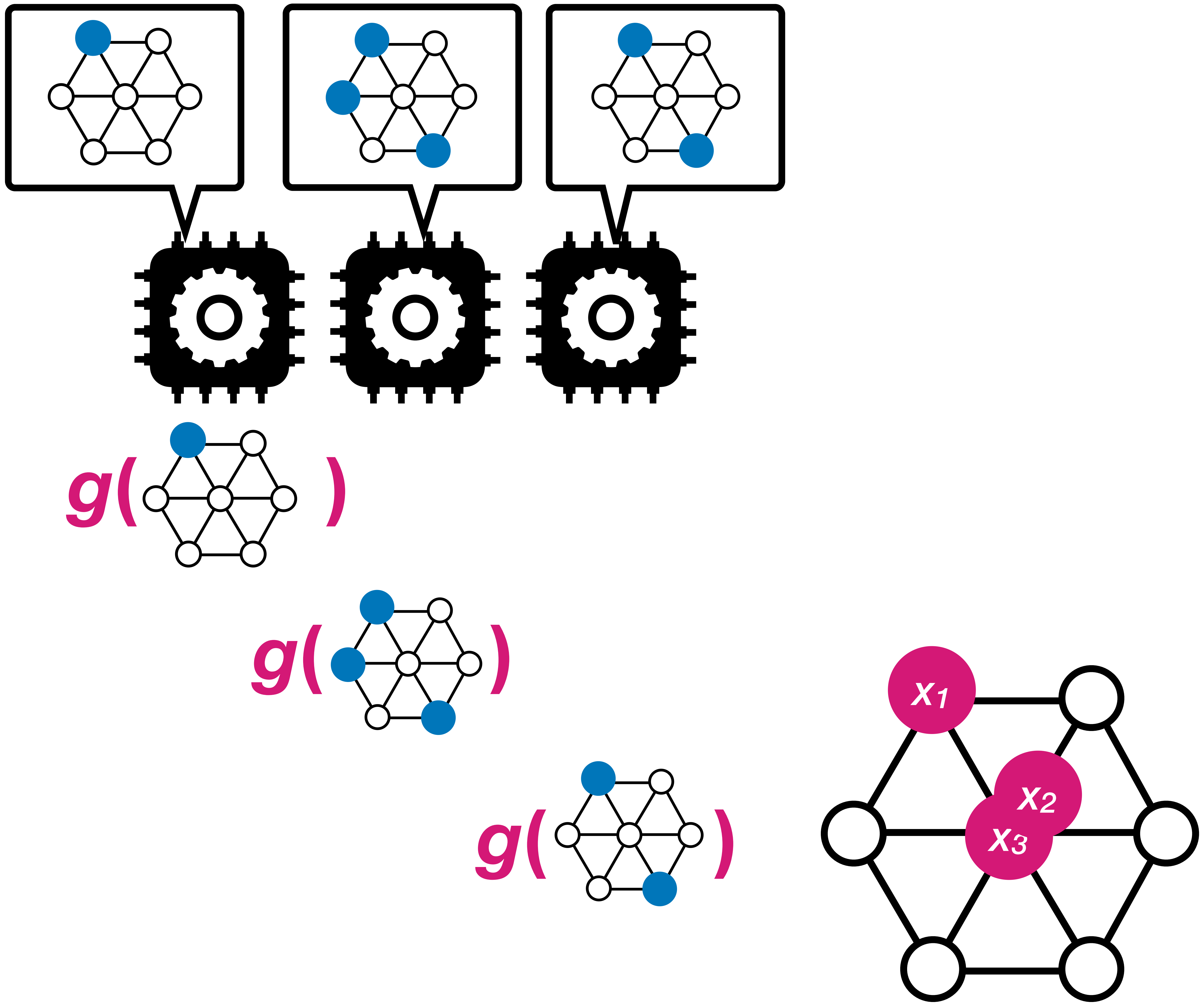


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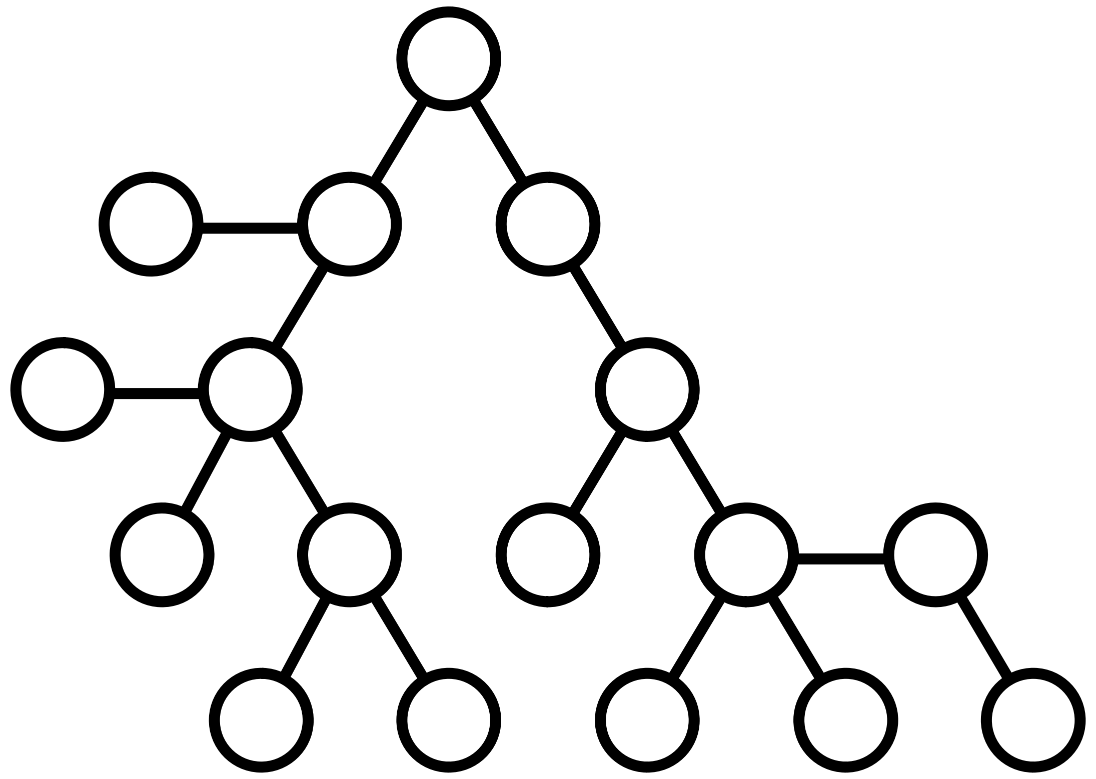
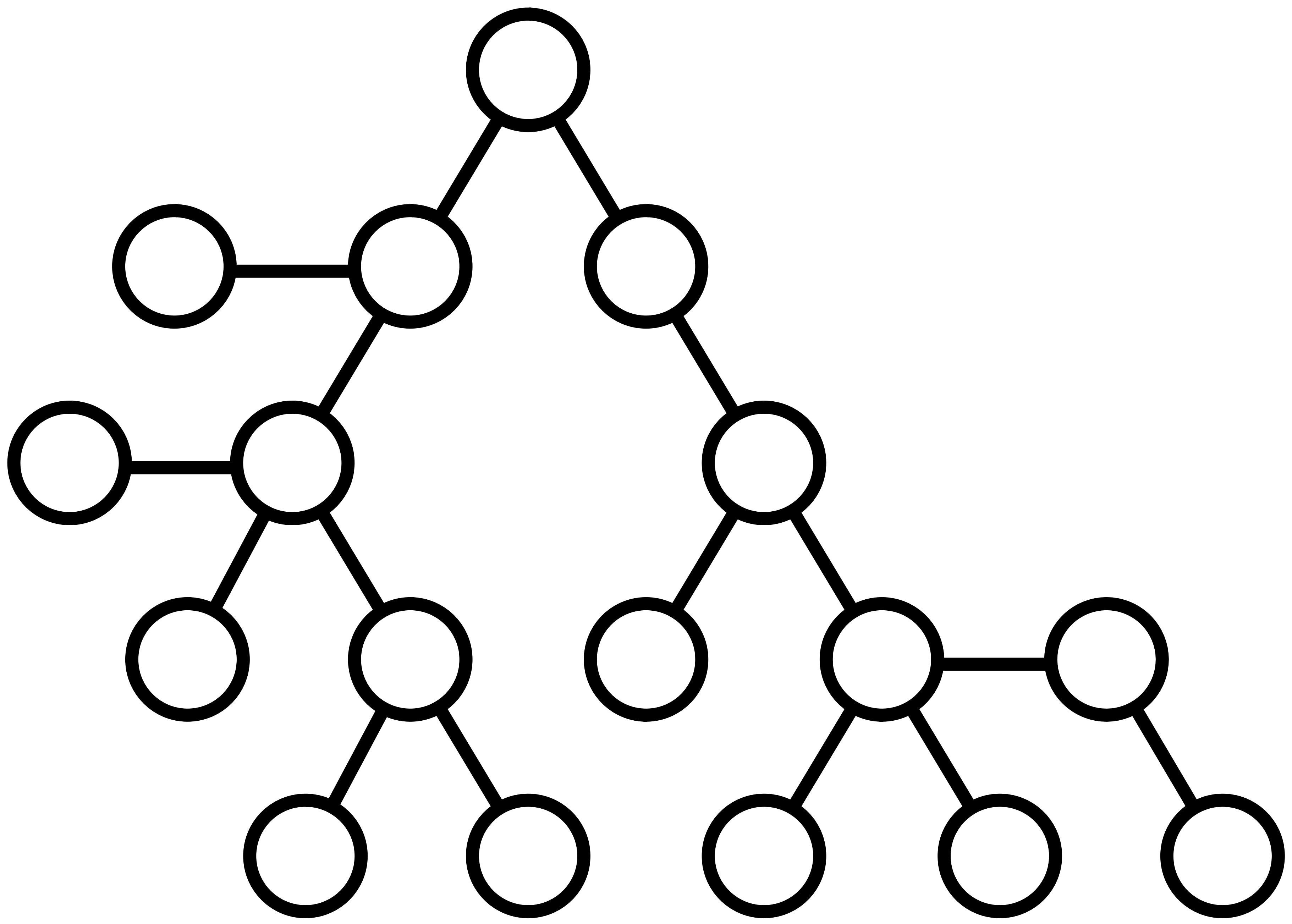
# A wait-free algorithm for trees

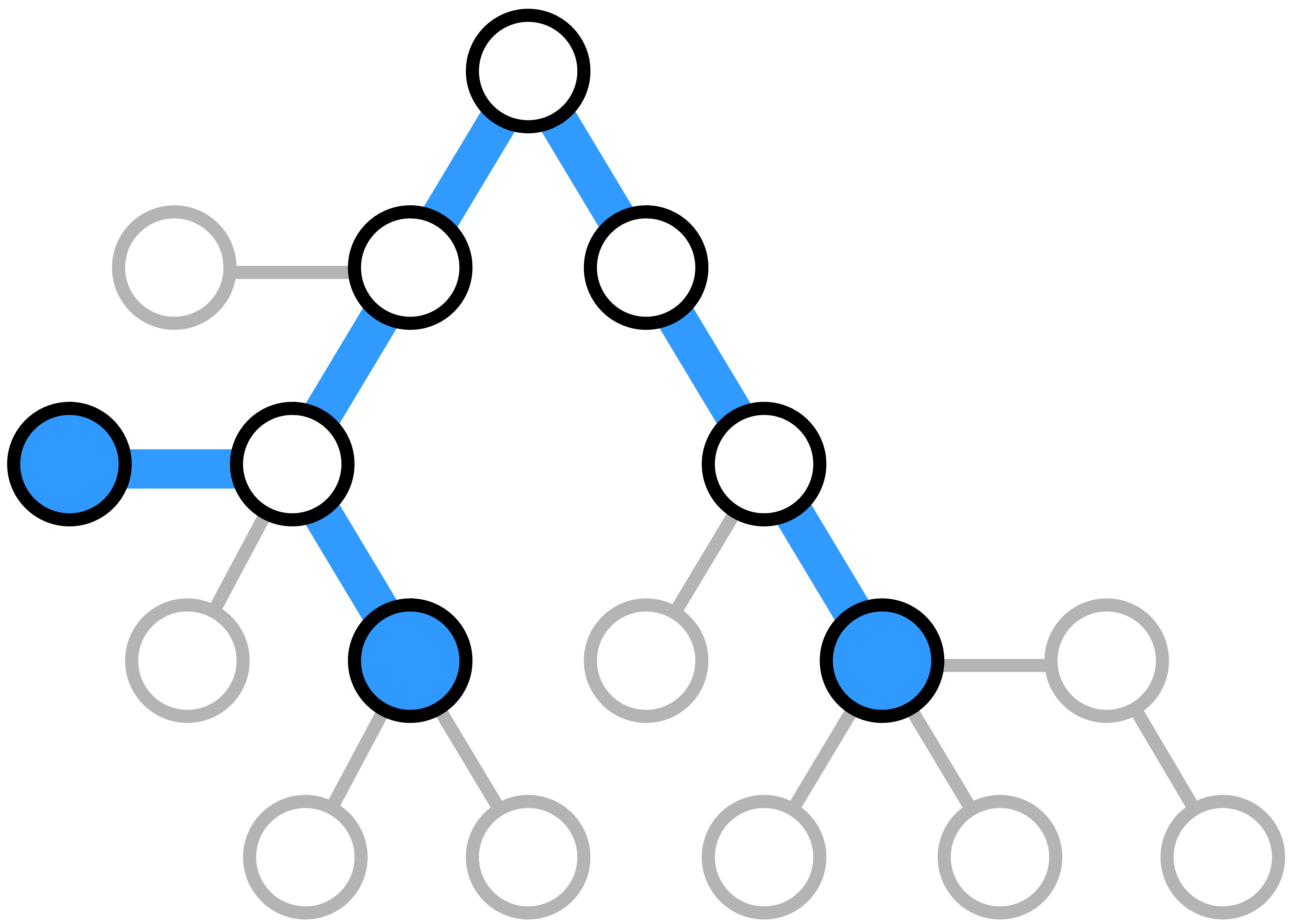


# Algorithm for trees



# Algorithm for trees

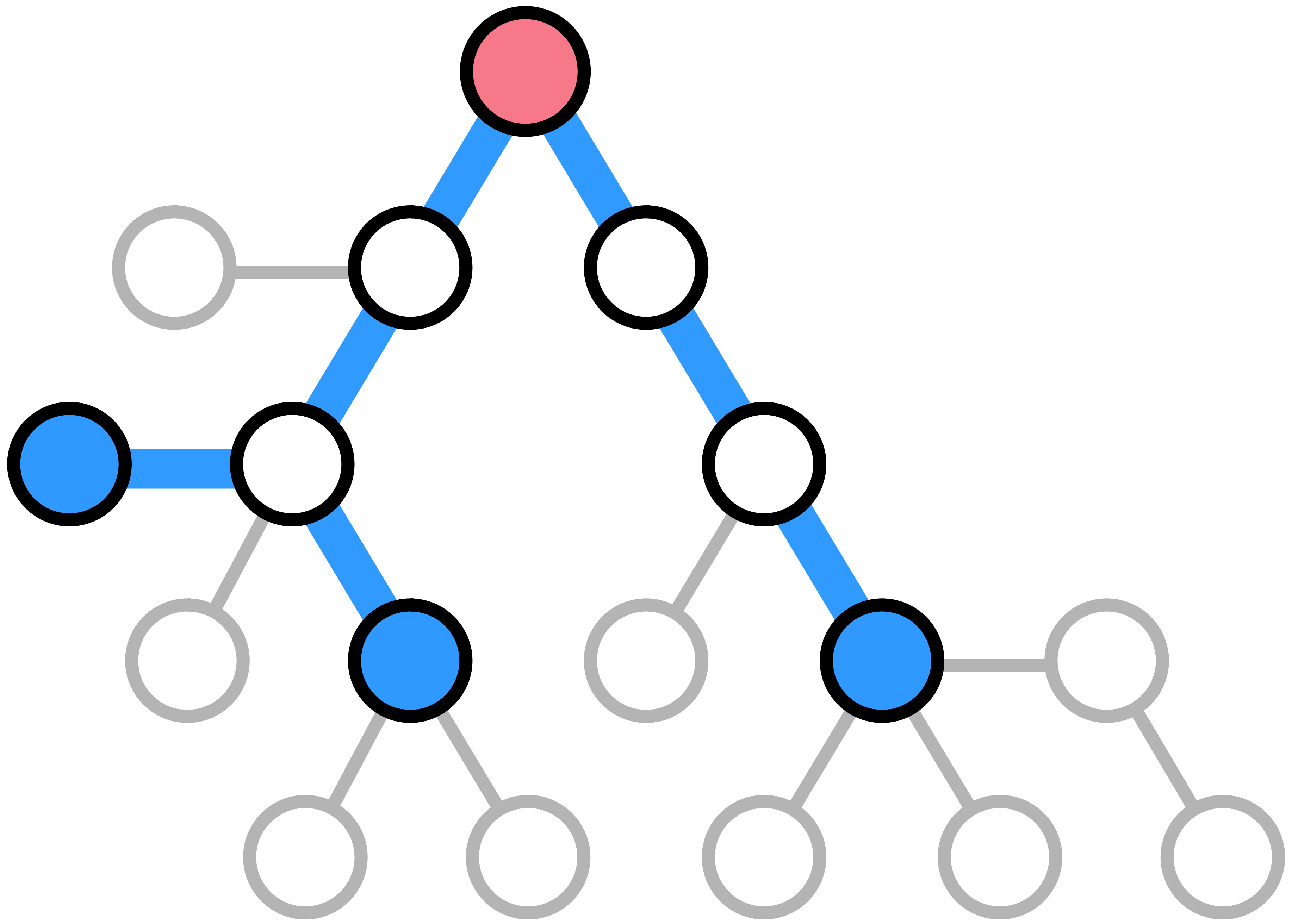
The ***convex hull*** **<** ***U*** **>** of ***U*** consists of all vertices on shortest paths between vertices of ***U*****.**



# Algorithm for trees

The ***convex hull*** **<** ***U*** **>** of ***U*** consists of all vertices on shortest paths between vertices of ***U.***

The ***center*** of **<** ***U*** **>** is the *set* of vertices that minimise the maximum distance to any other node in **<** ***U*** **>.**



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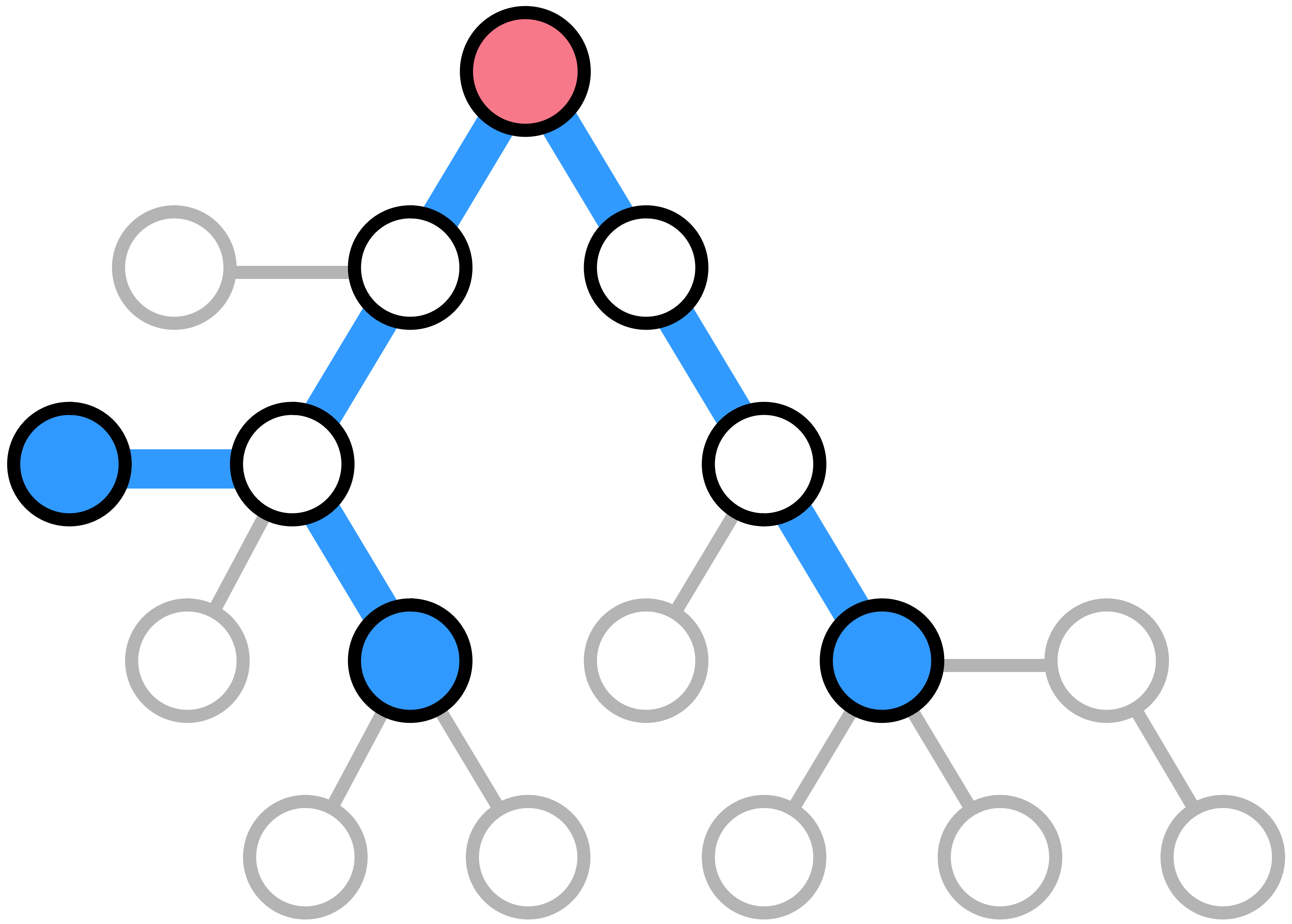
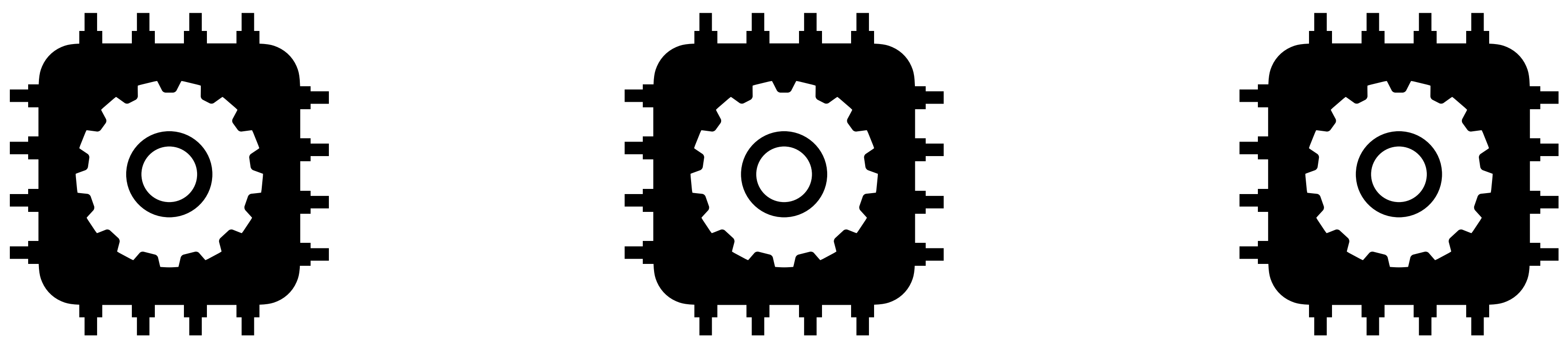


**Update rule:**
let ***g*( *U* )** to be a vertex in the *center* of **< *U* >**

# Algorithm for trees



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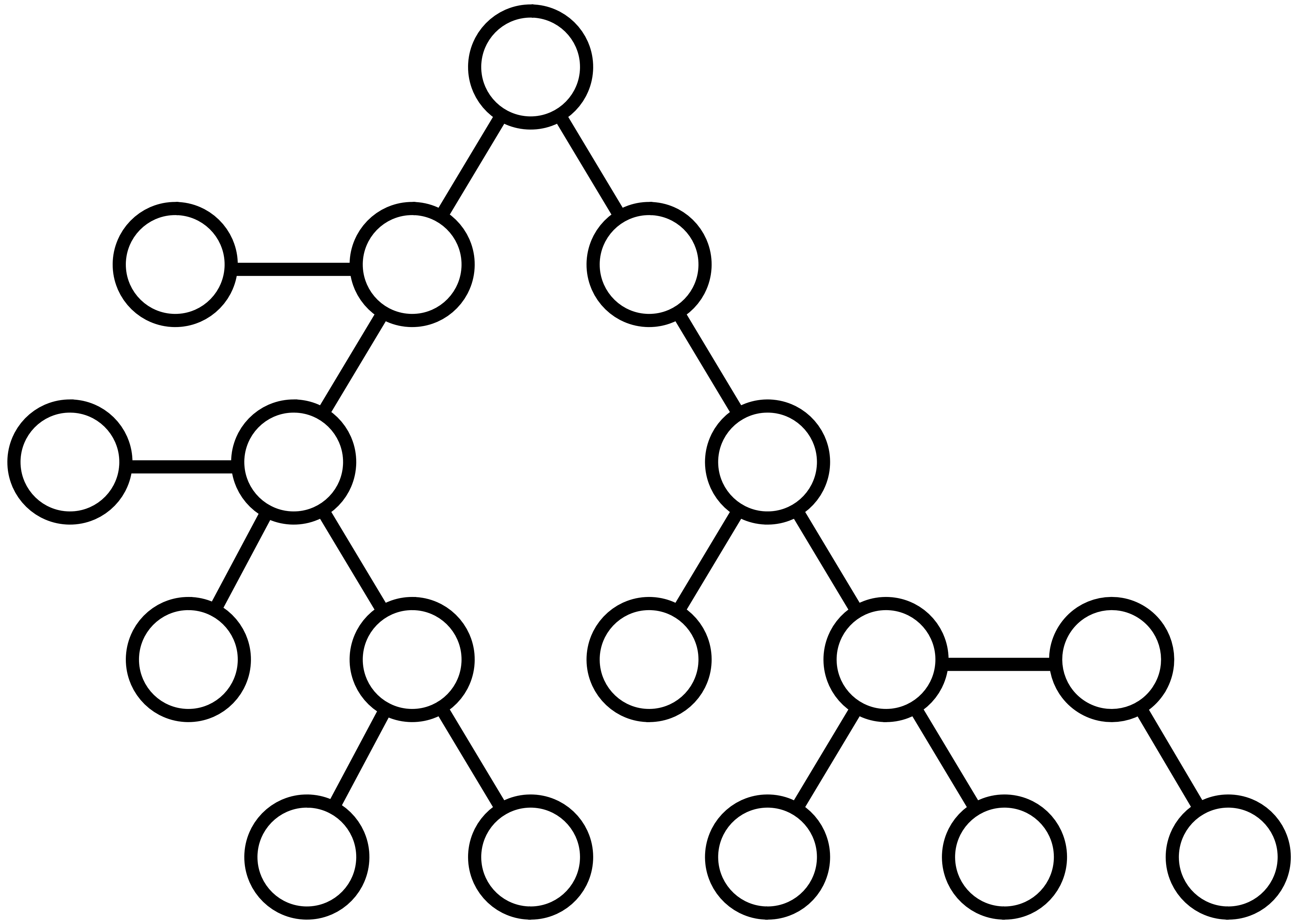
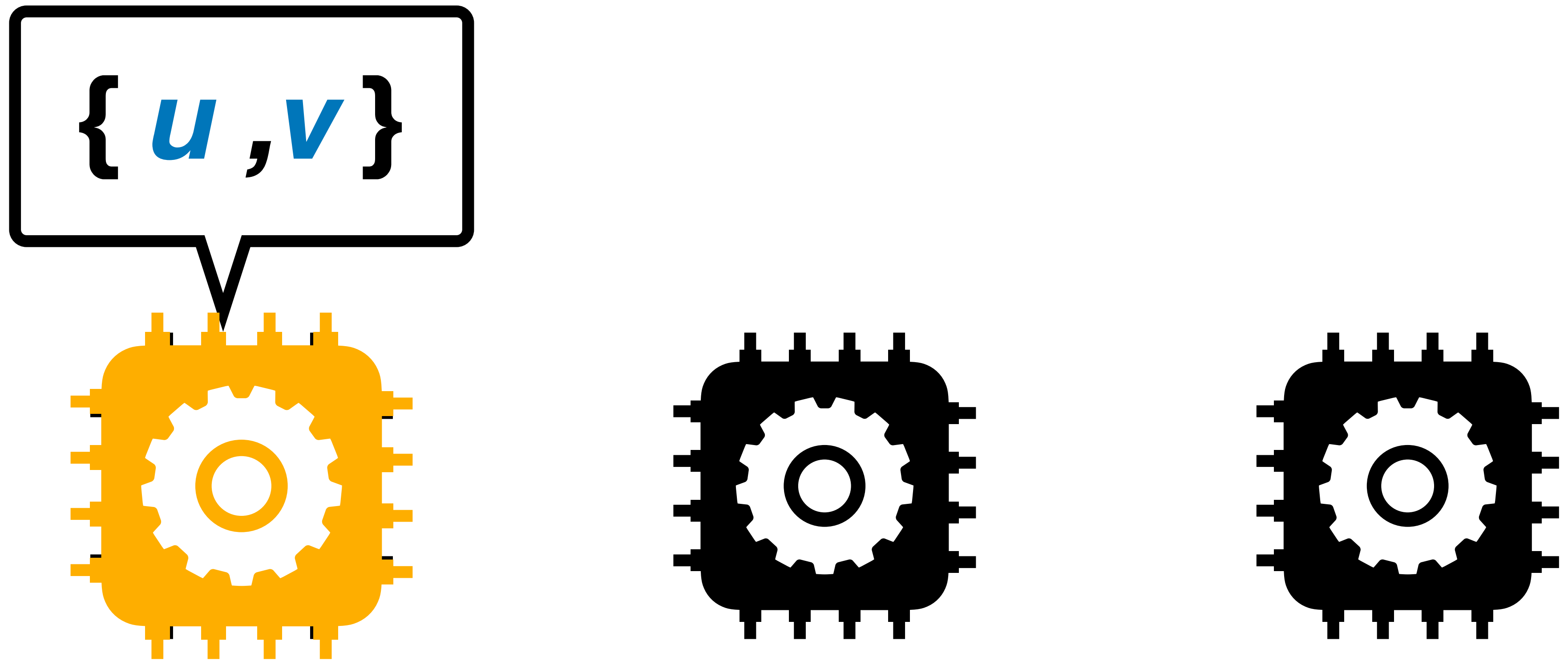
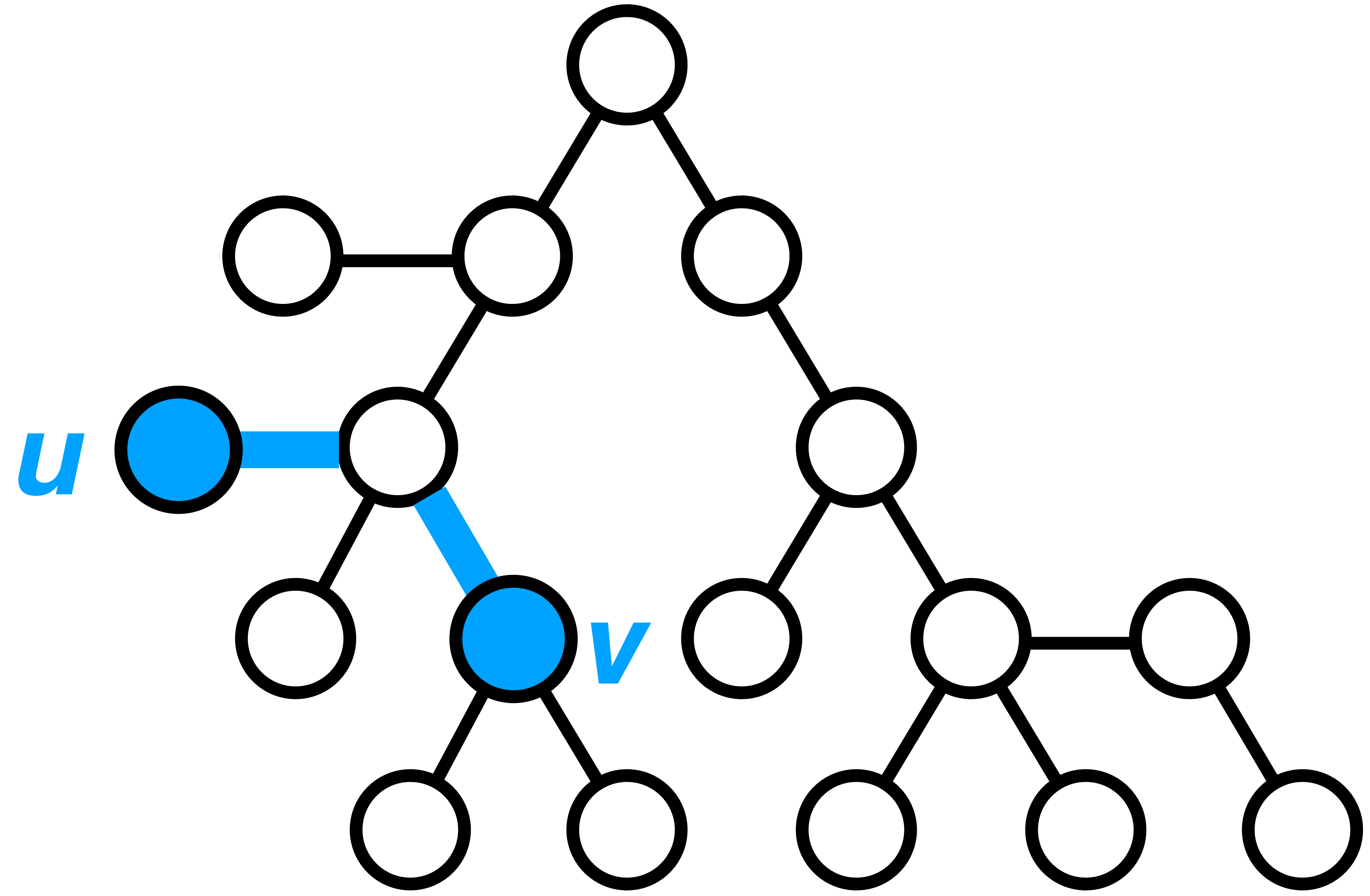


# Algorithm for trees



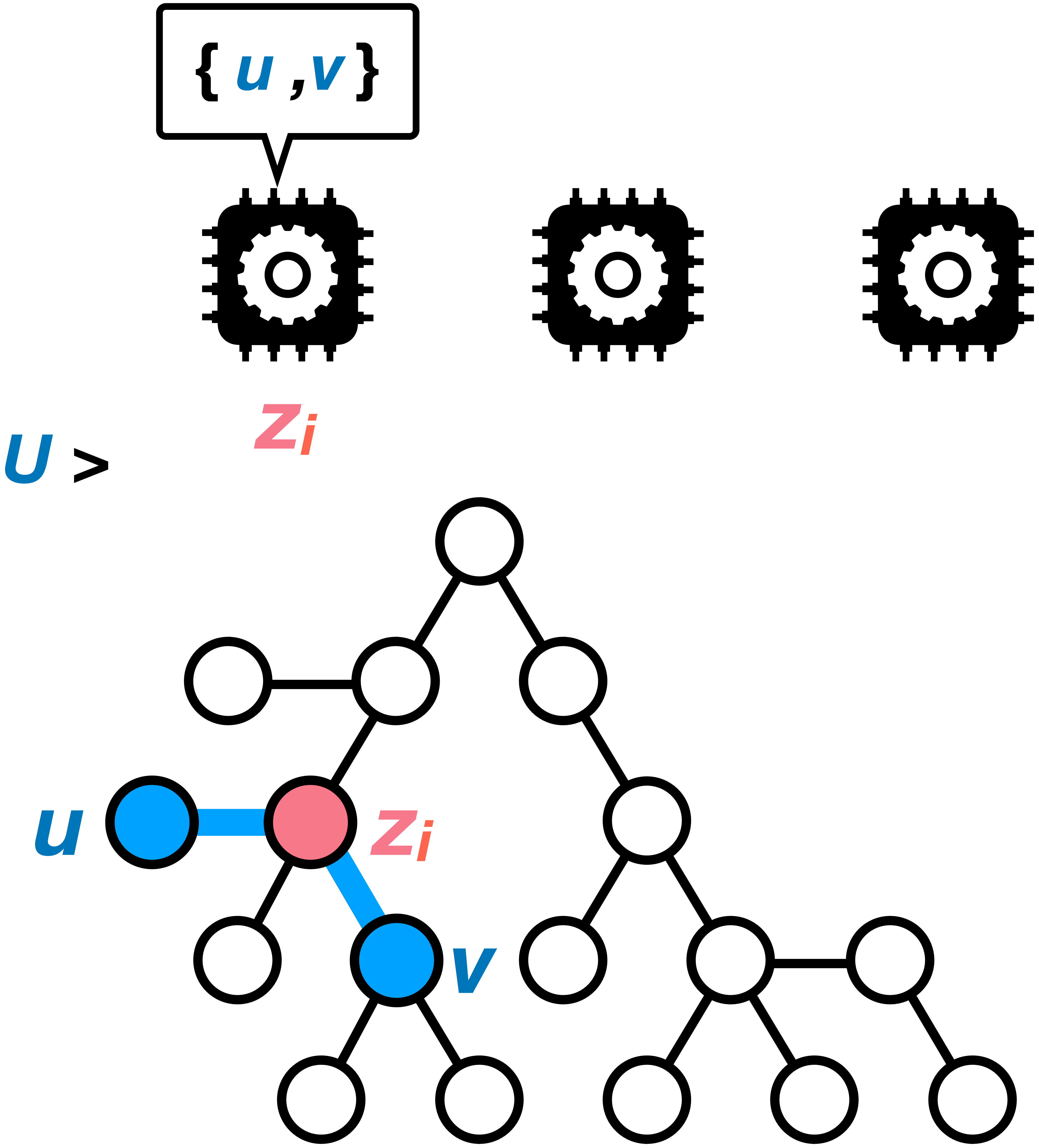

**Update rule:**

let *g( U )* to be a vertex in the *center* of $< U >$



# Algorithm for trees

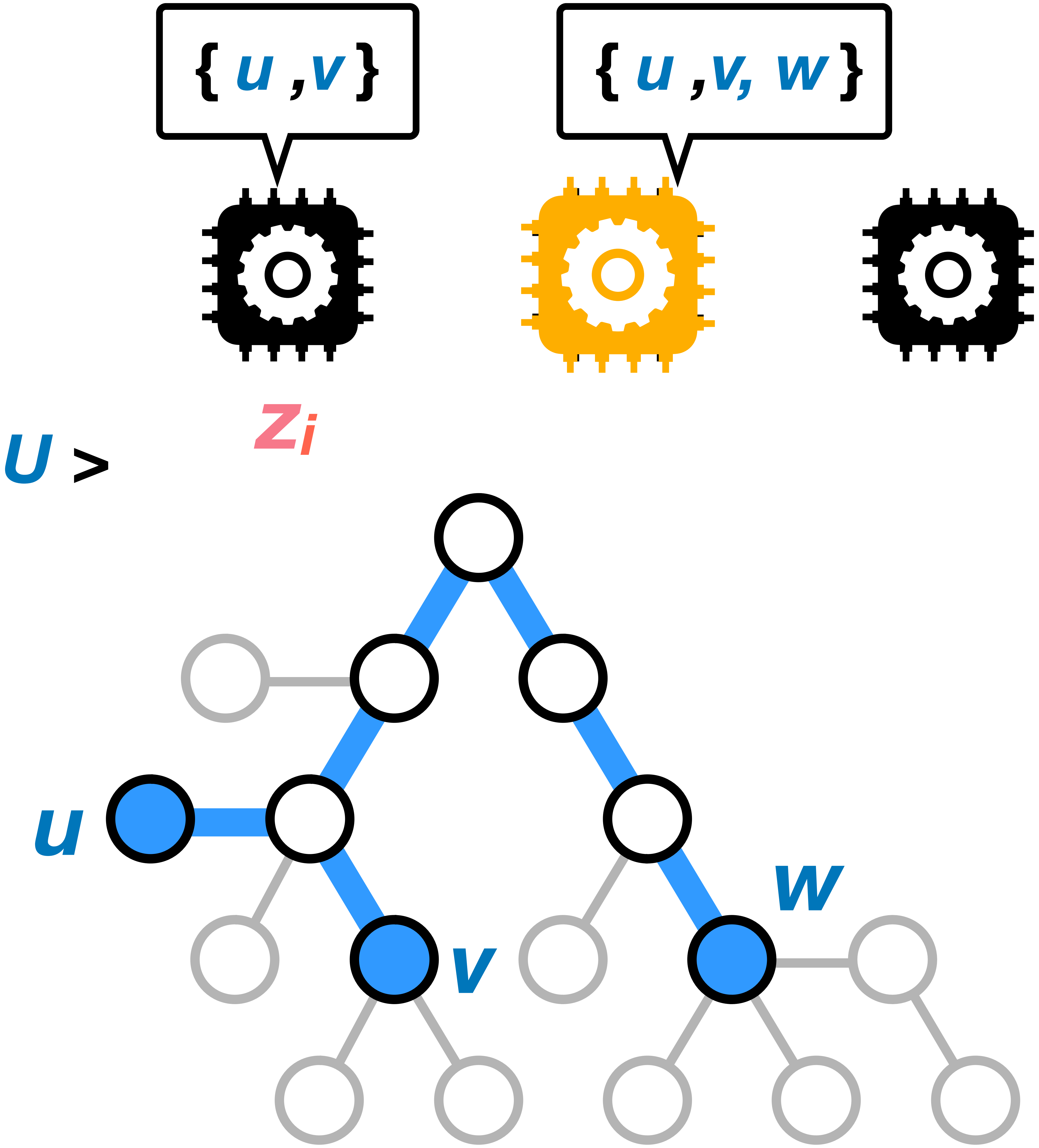
**Update rule:**
let $g(U)$ to be a vertex in the *center* of $< U >$



# Algorithm for trees

**Update rule:**
let $g(U)$ to be a vertex in the *center* of $< U >$

**Containment property:**
$V_{j(1)} \subseteq V_{j(2)} \subseteq \ldots \subseteq V_{j(n)}$



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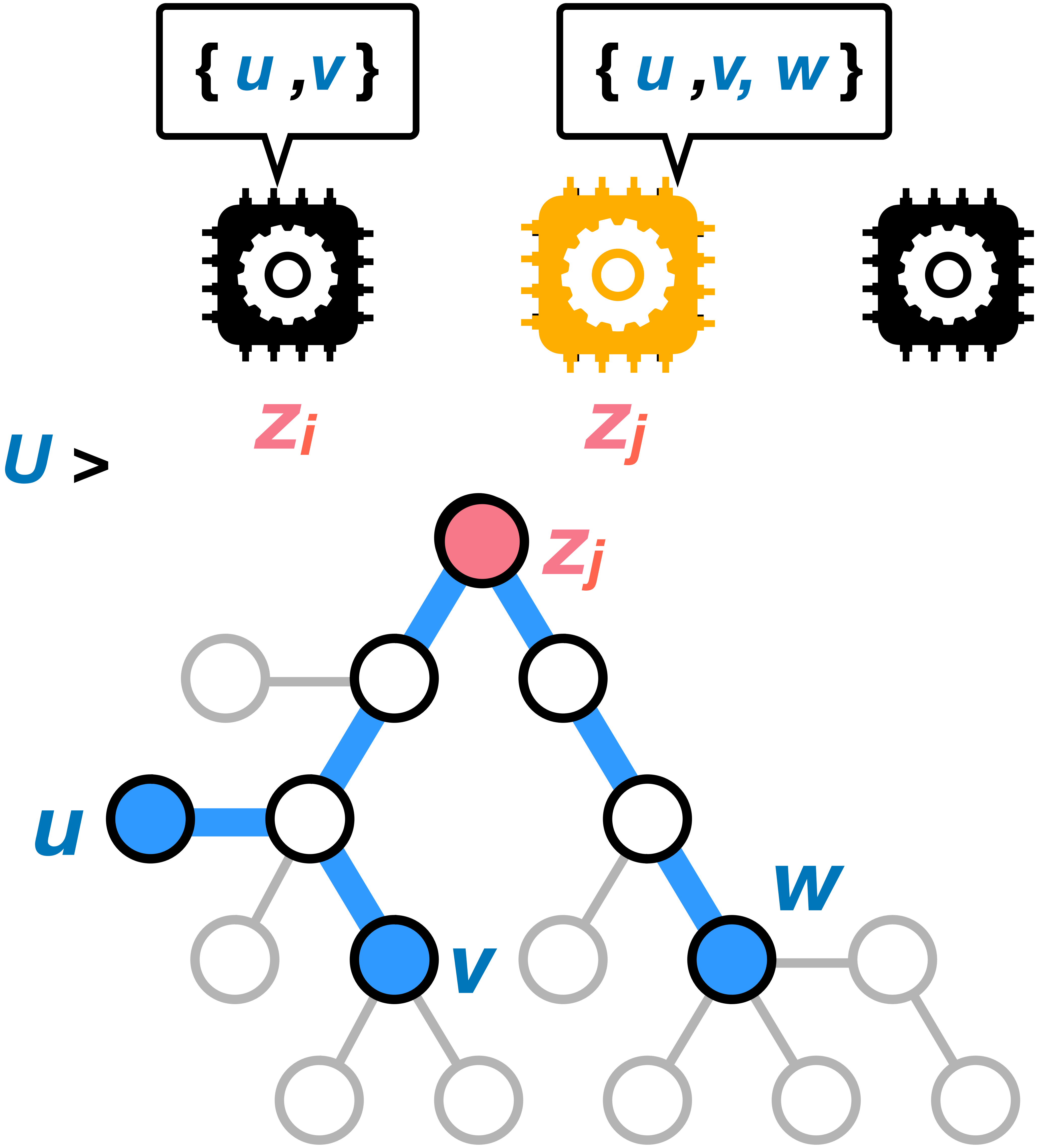

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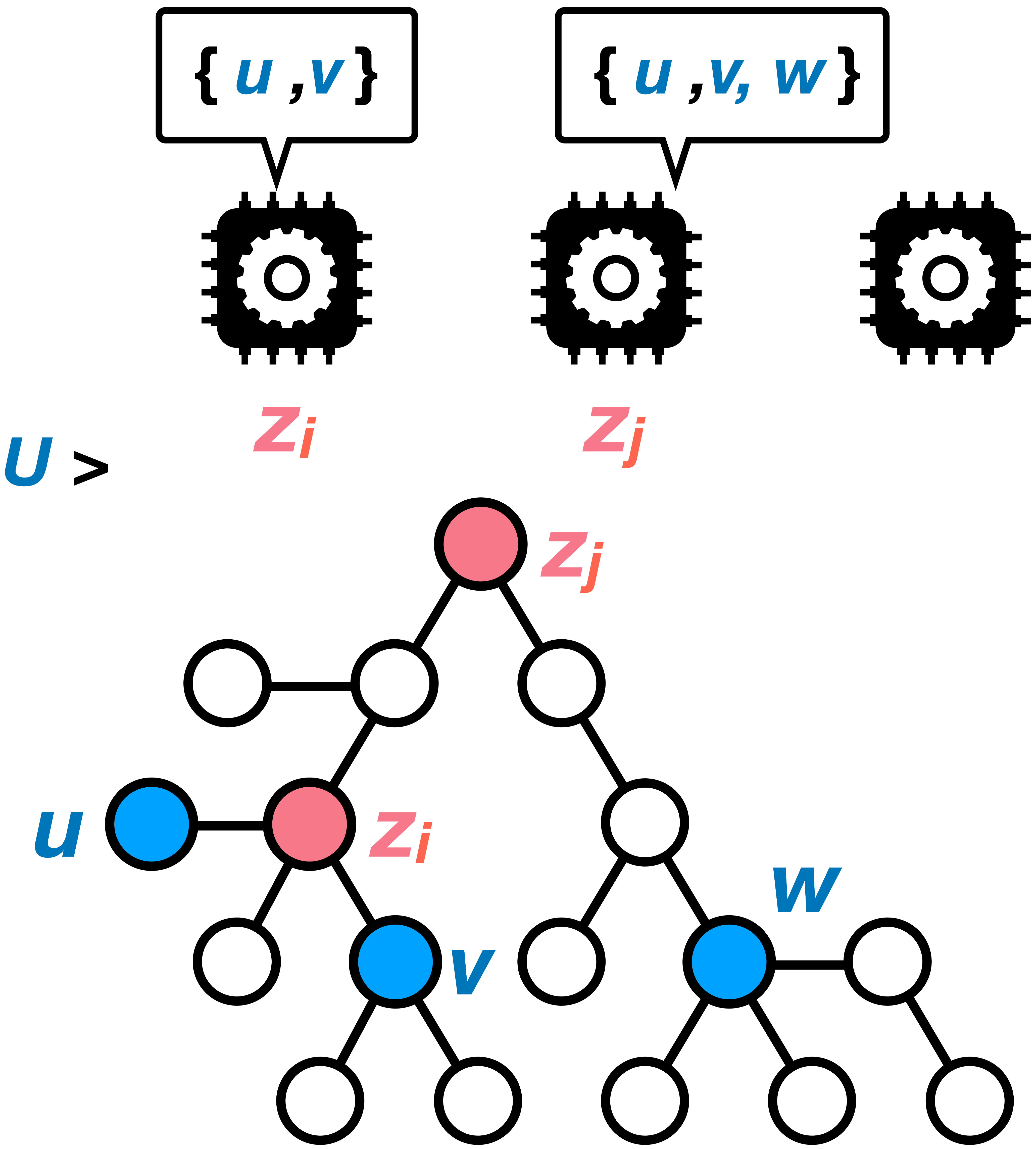
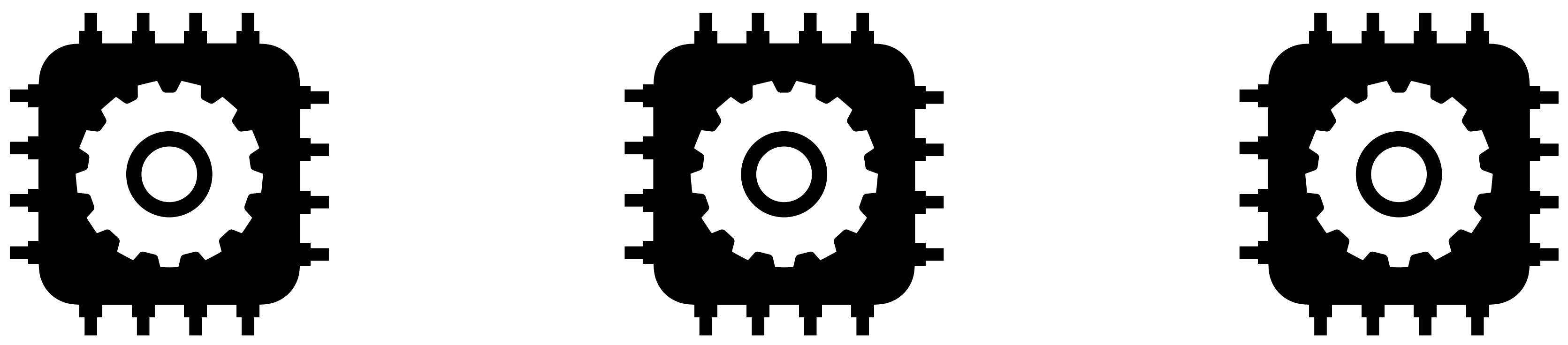
Distance is now (roughly) half of diameter of $< X >$.



# Algorithm for trees



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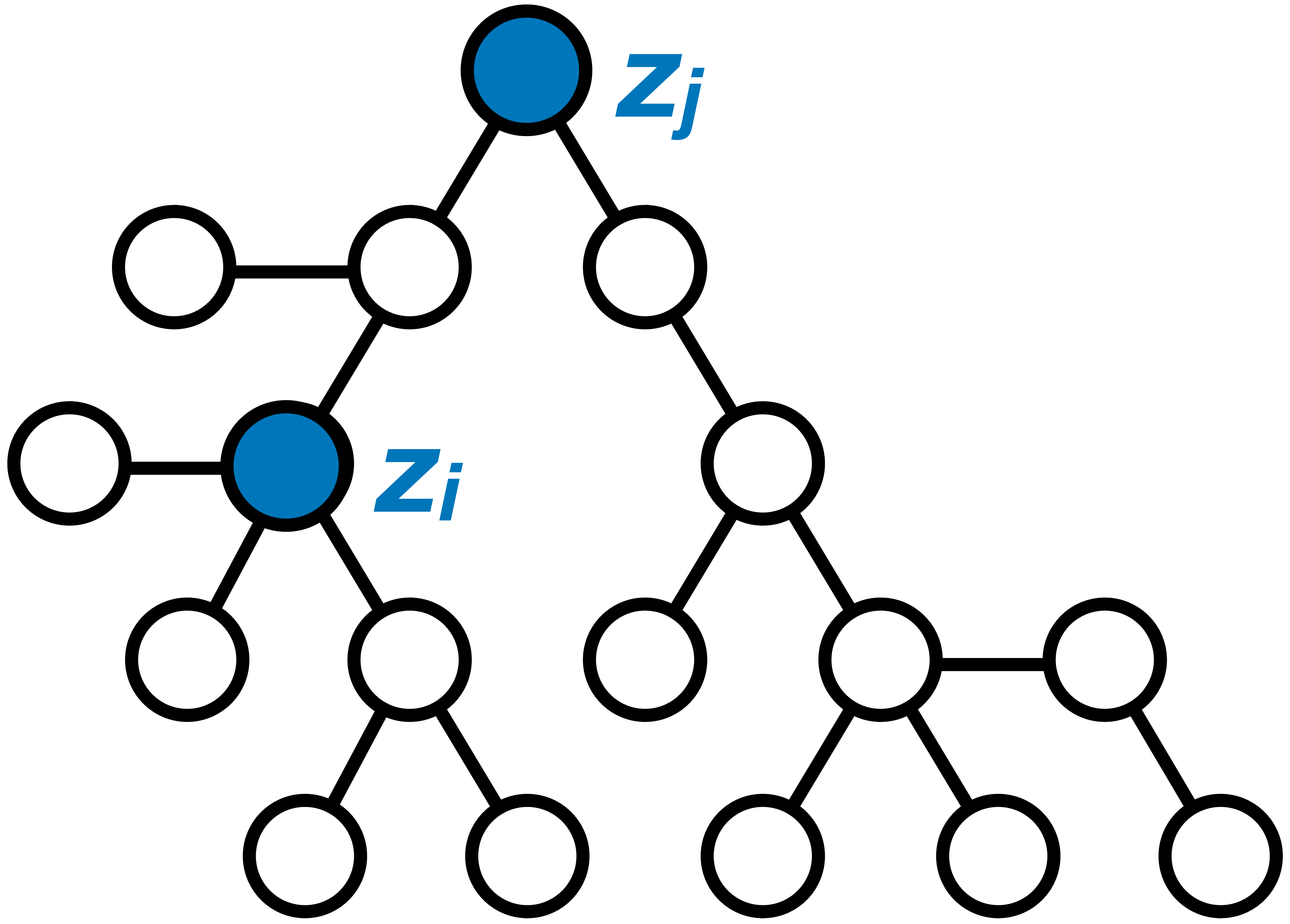

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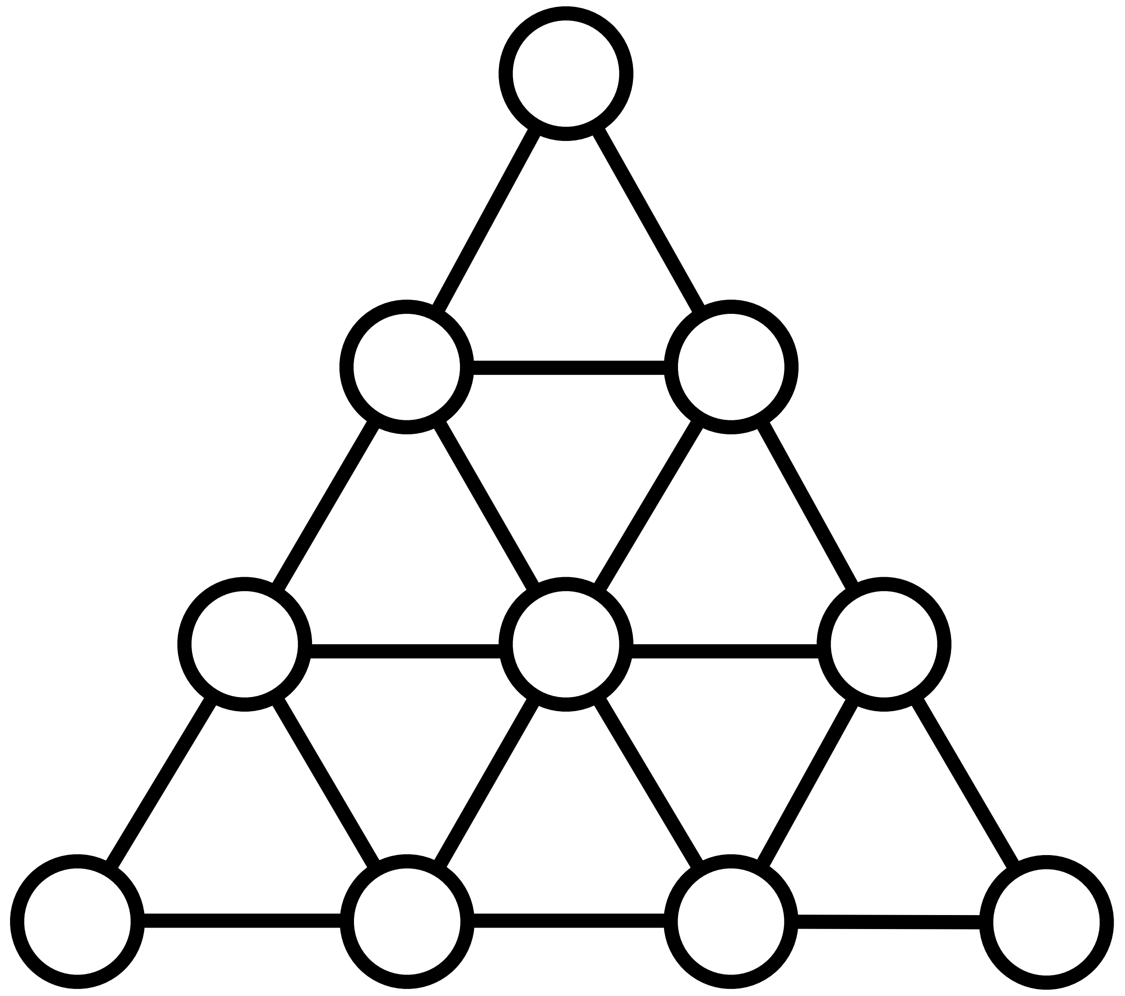
Same idea extends:
— **chordal graphs:** radius ≈ 1/2 · diameter
— **bridged* graphs:** radius ≈ 2/3 · diameter



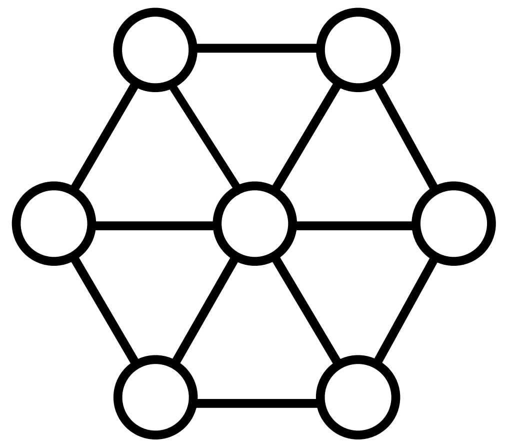
# Wait-free solvability in other graph classes

— **Paths**

e.g, Biran, Moran, Zaks (1990); Attiya, Lynch, Shavit (1994); Schenk (1995)

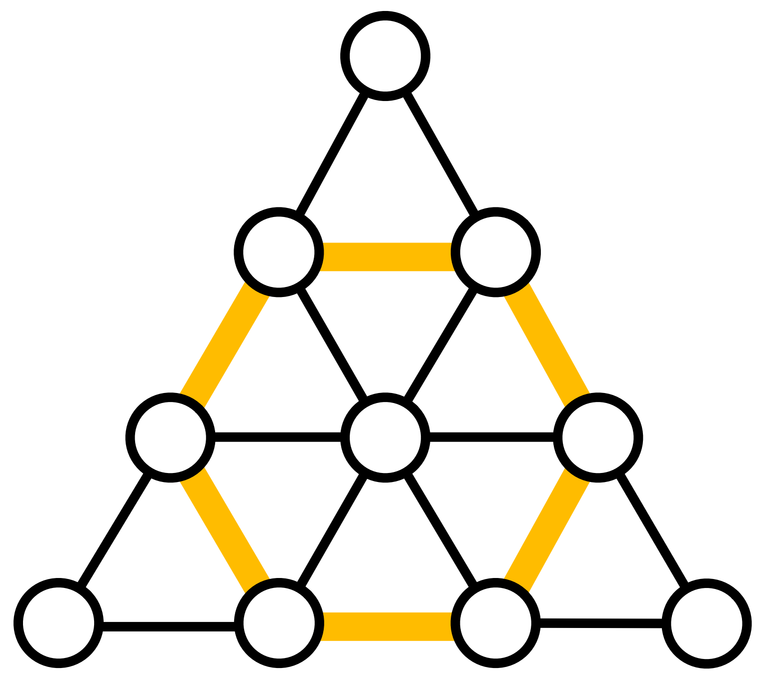
— **Clique graph is a tree or has radius one**

Alcántara, Castañeda, Flores-Peñaloza, Rajsbaum (2019)



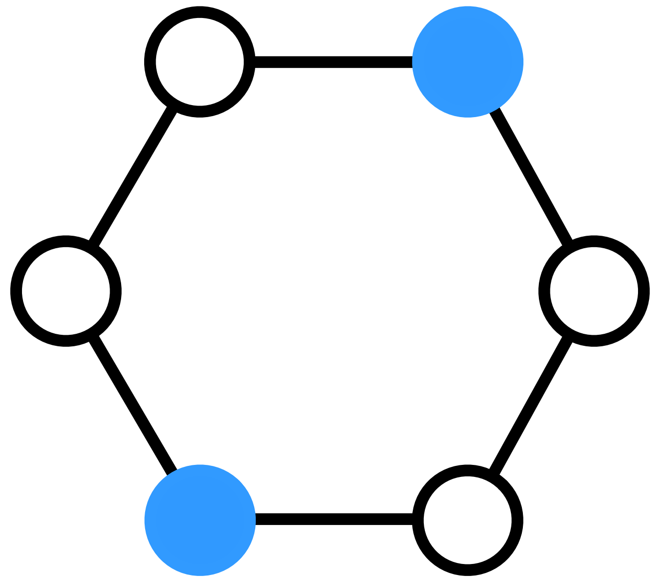
— **Nicely bridged graphs (contains all chordal graphs)**

Alistarh, Ellen, Rybicki (2023)



# **Impossibility results**
# for wait-free algorithms

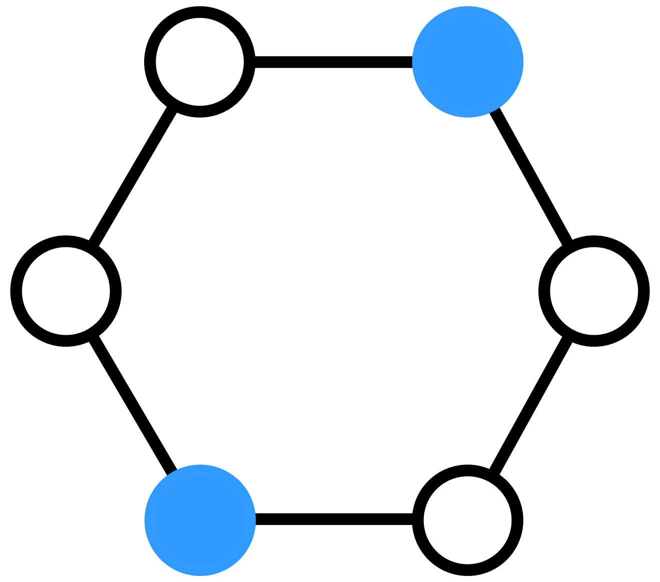
# Impossibility on cycles



**Theorem.** There is **no wait-free algorithm** for ***n* > 2** processes that solves approximate agreement on cycles of length at least 4.

Castañeda, Rajsbaum, and Roy (2018)

# Impossibility on cycles

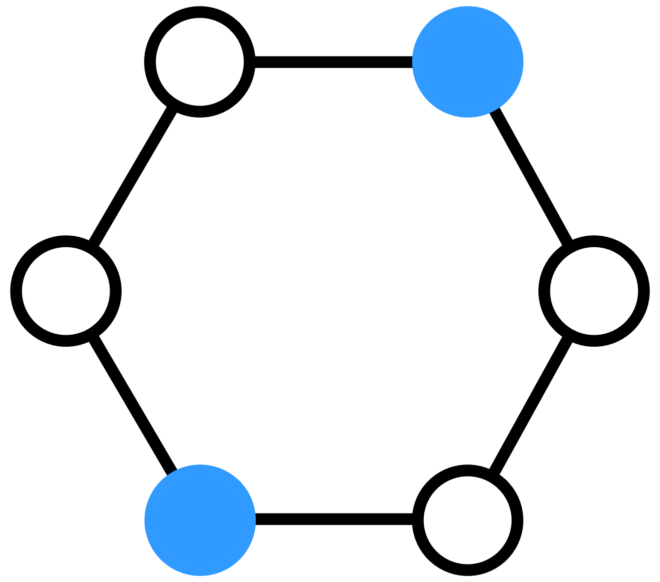


**Theorem.** There is **no wait-free algorithm** for $\boldsymbol{n > 2}$ processes that solves approximate agreement on cycles of length at least 4.

**Corollary.** Any $f$-resilient **synchronous message-passing algorithm** requires at least $\lfloor f/2 \rfloor + 1$ rounds.

**Proof:** Apply BG simulation + Gafni's round-by-round fault-detectors.

# Impossibility on cycles



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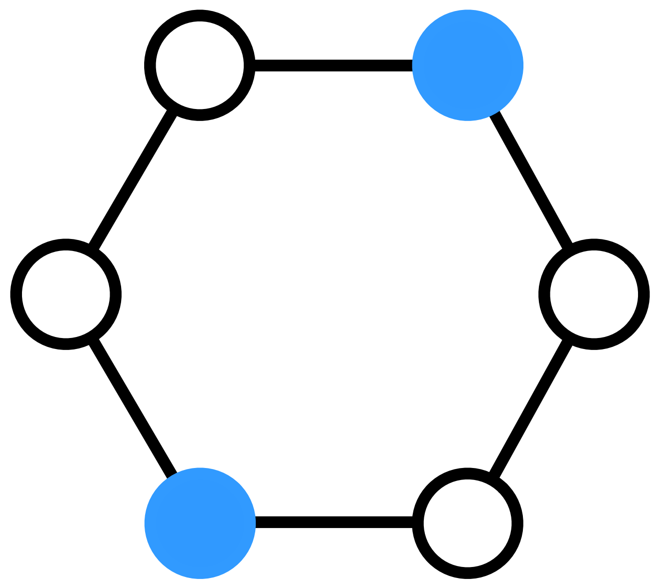
**Two flavours of proofs exist:**

— **Reductions** from 2-set agreement
e.g., Castañeda, Rajsbaum, and Roy (2018)

— **Topological proofs** using variants of Sperner's lemma
e.g., Alistarh, Ellen, Rybicki (2023) and Liu (2022)

# Impossibility on cycles



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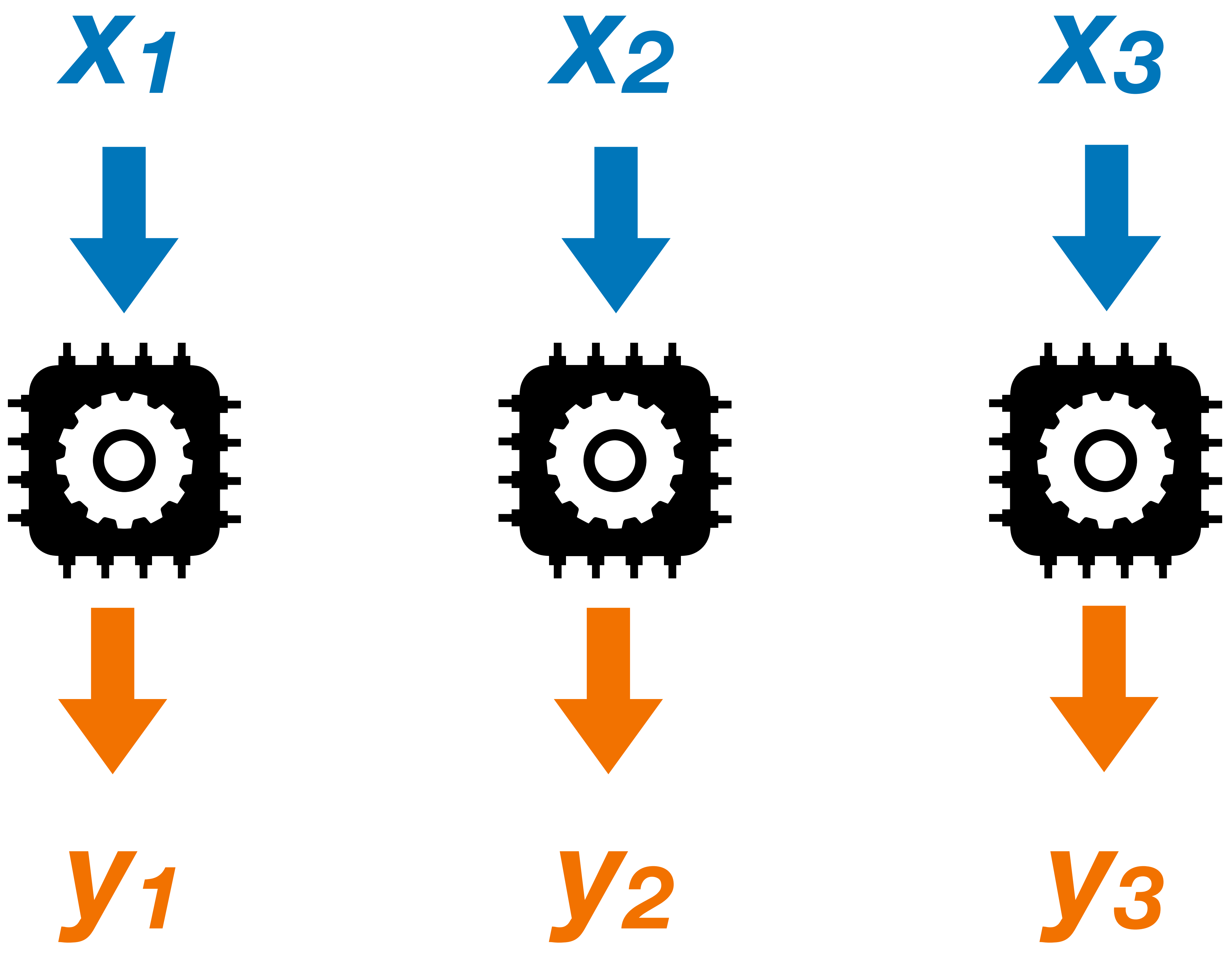
# **A hard problem:** 2-set agreement

$X \subseteq \{ 1, 2, 3 \}$

Constraints:

— **validity:** $Y \subseteq X$

— **agreement:** $| Y | \leq 2$.



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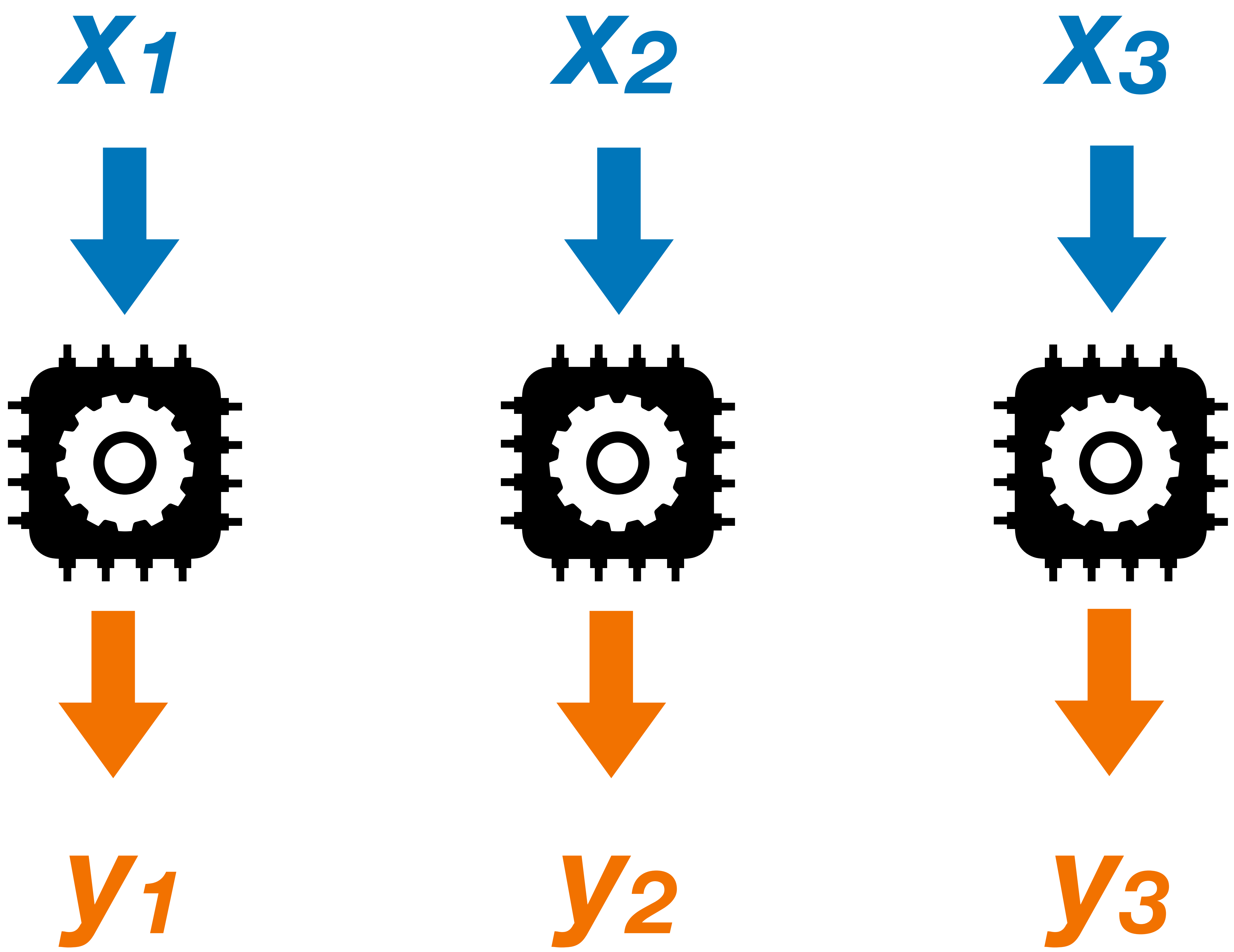



**Theorem.**

There is no ***wait-free*** algorithm for 2-set agreement for n > 2.

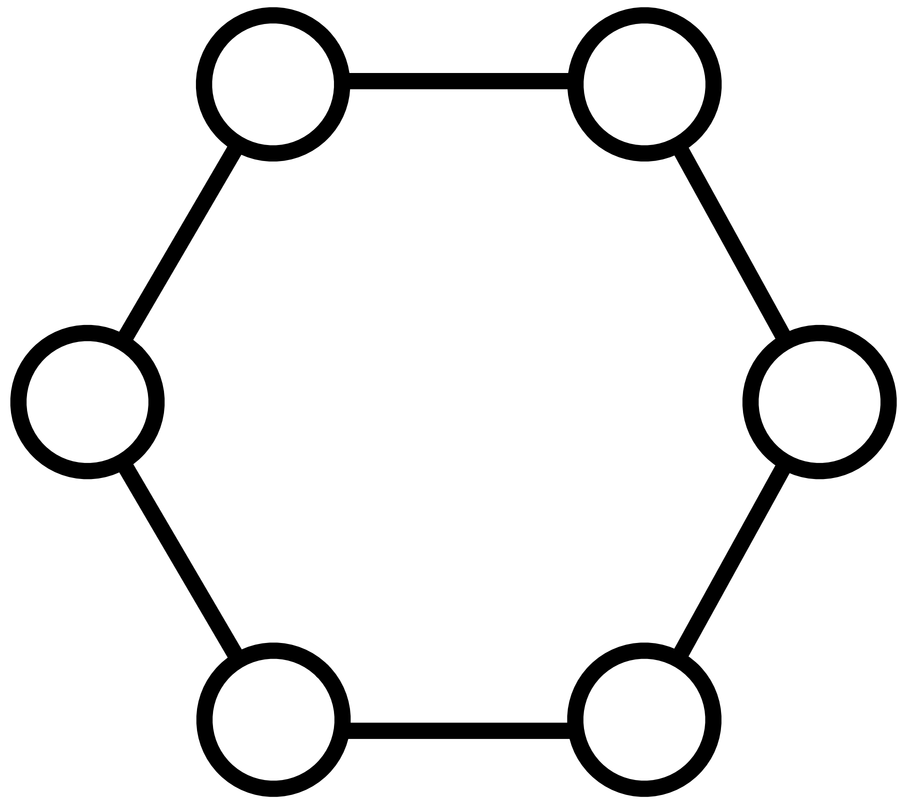
Borowsky and Gafni (1993), Herlihy and Shavit (1999), Saks and Zaharoglou (2000)

# A reduction from 2-set agreement

## Idea:

Suppose there is a wait-free algorithm that solves approximate agreement on ***C***.

Then we can solve 2-set agreement.



***C***

# A reduction from 2-set agreement

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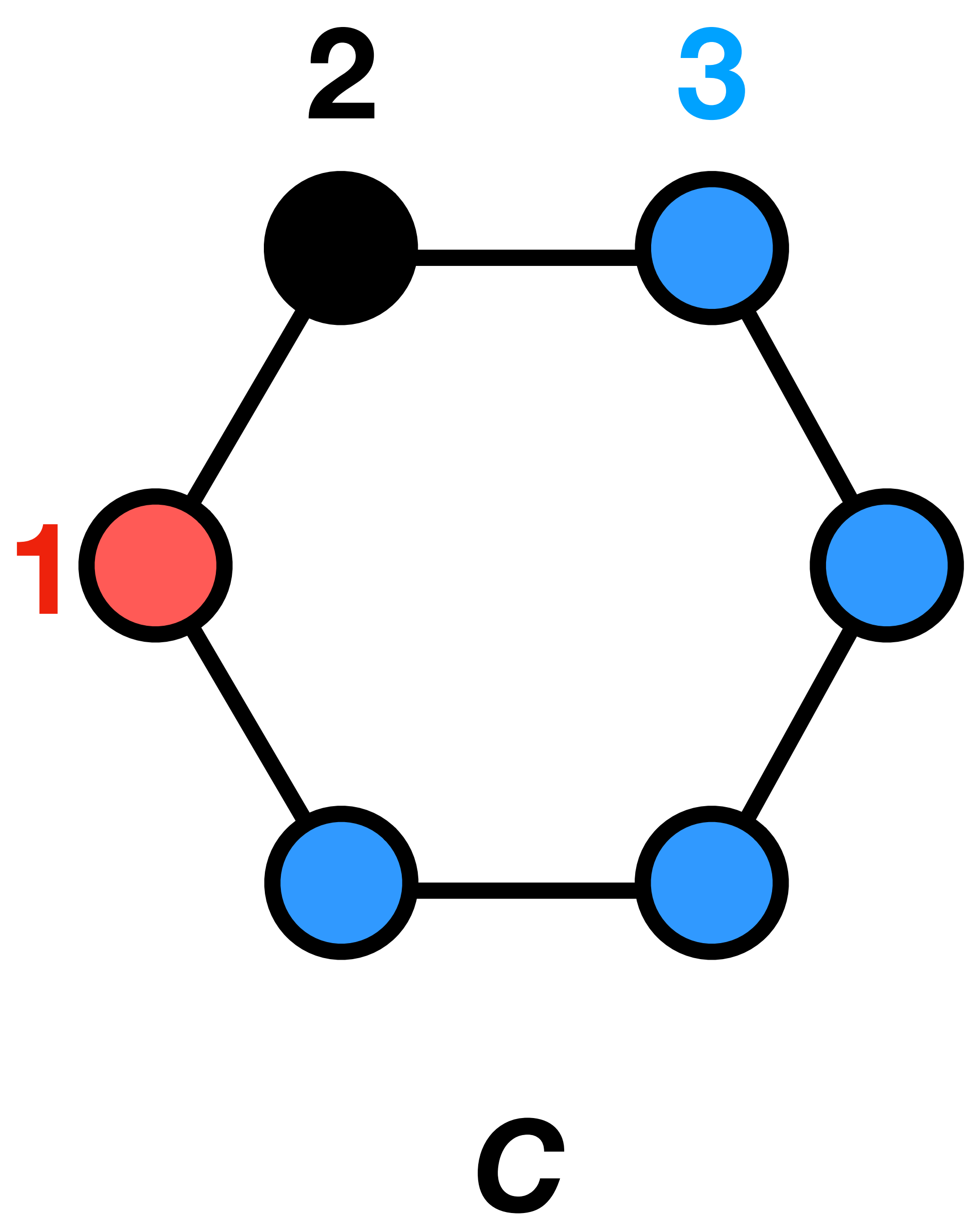
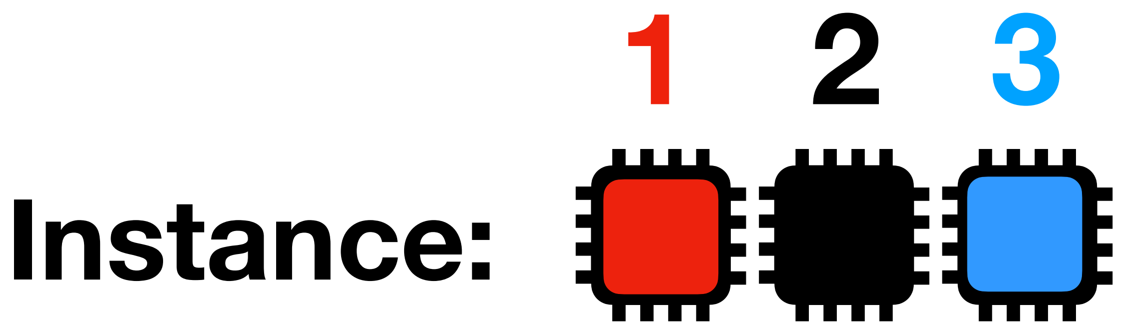
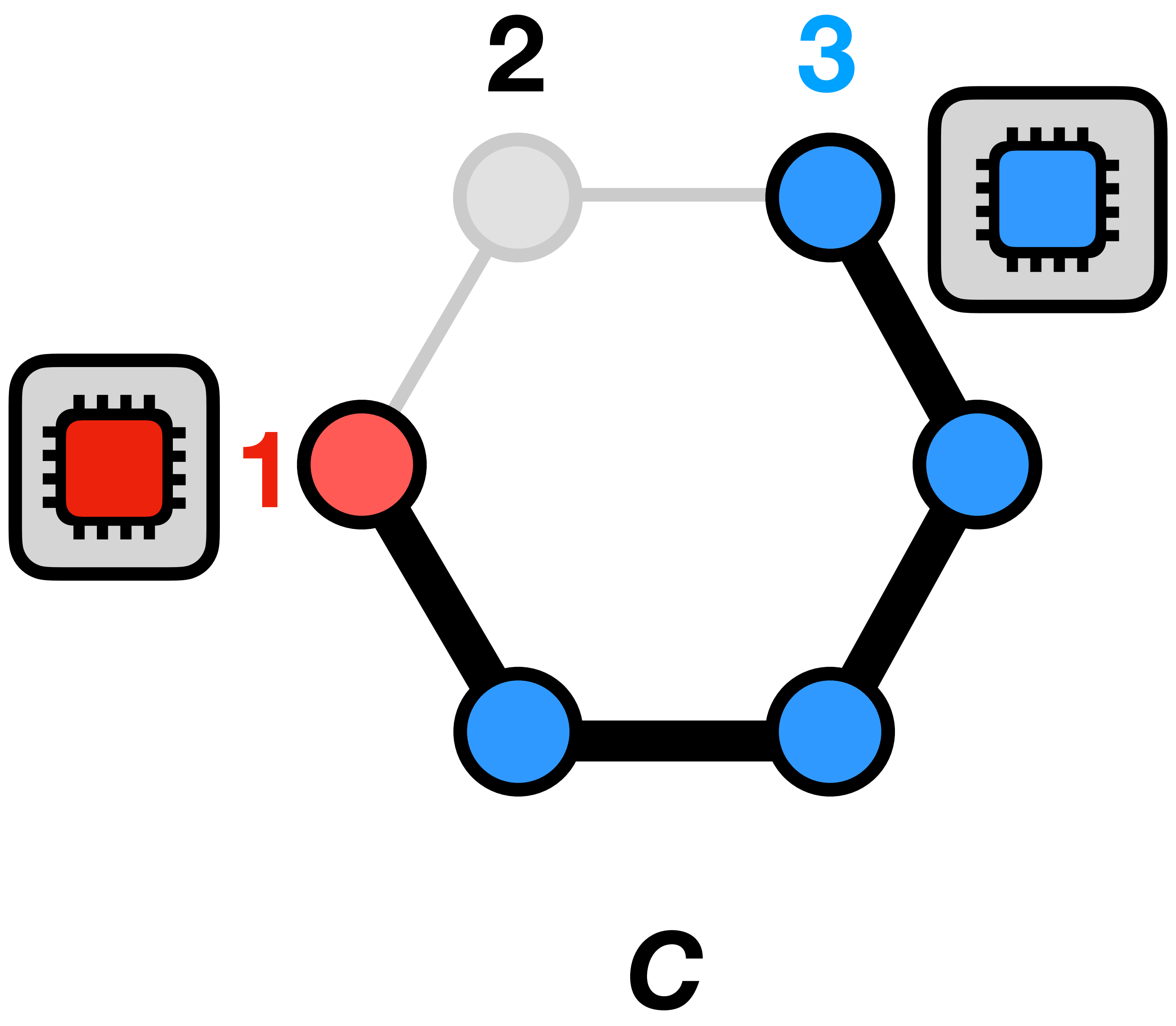


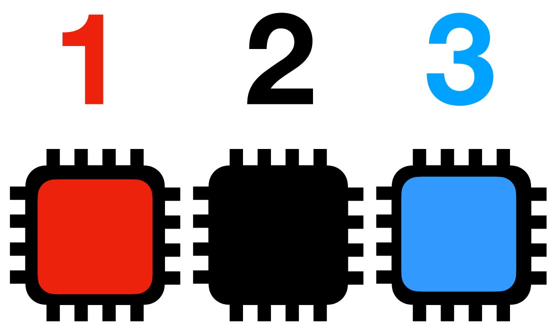
# A reduction from 2-set agreement

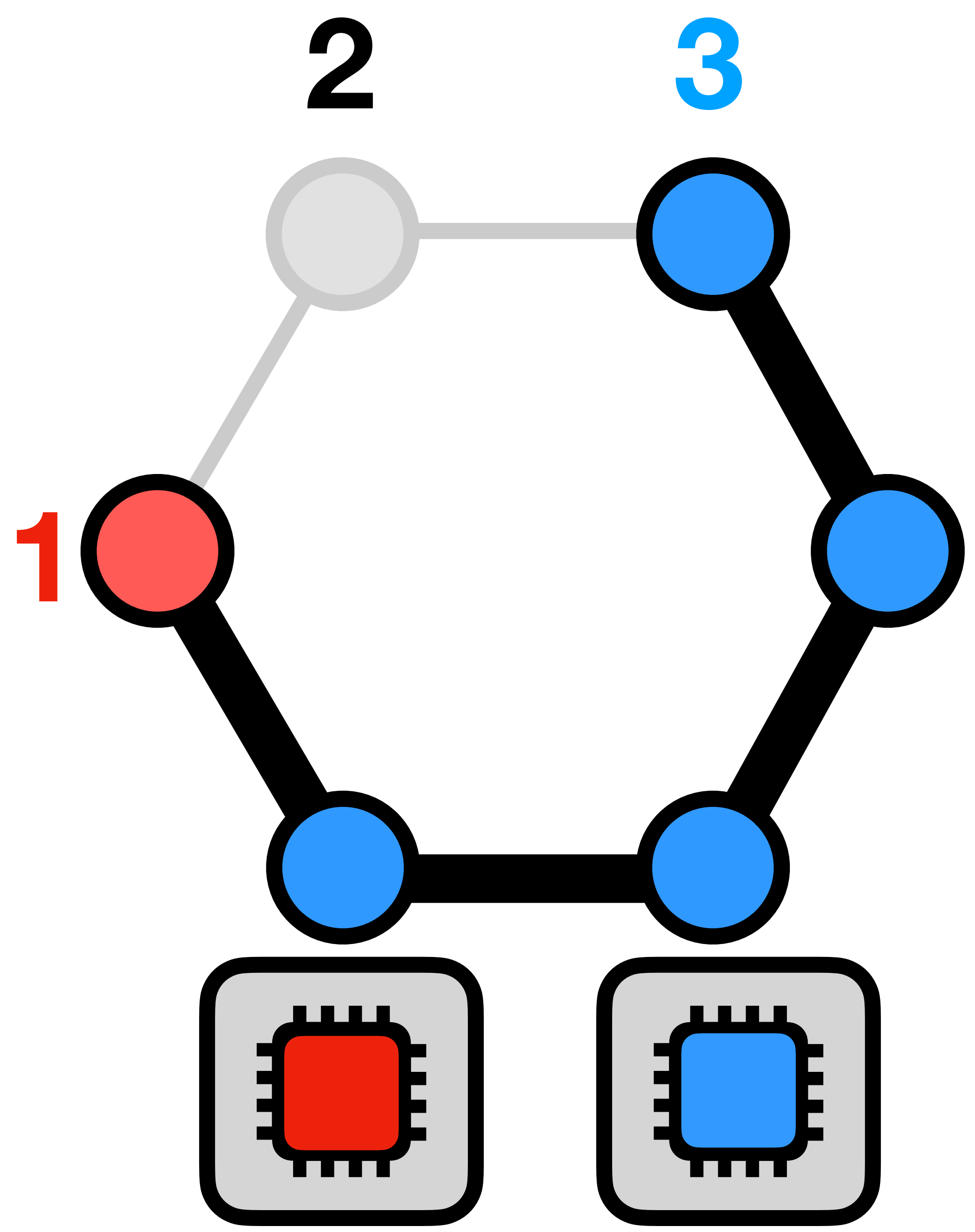



1. Processes with inputs **{ 1, 3 }** run a wait-free algorithm for approximate agreement on the **path** obtained by removing the black vertex **2**.



# A reduction from 2-set agreement

Instance: 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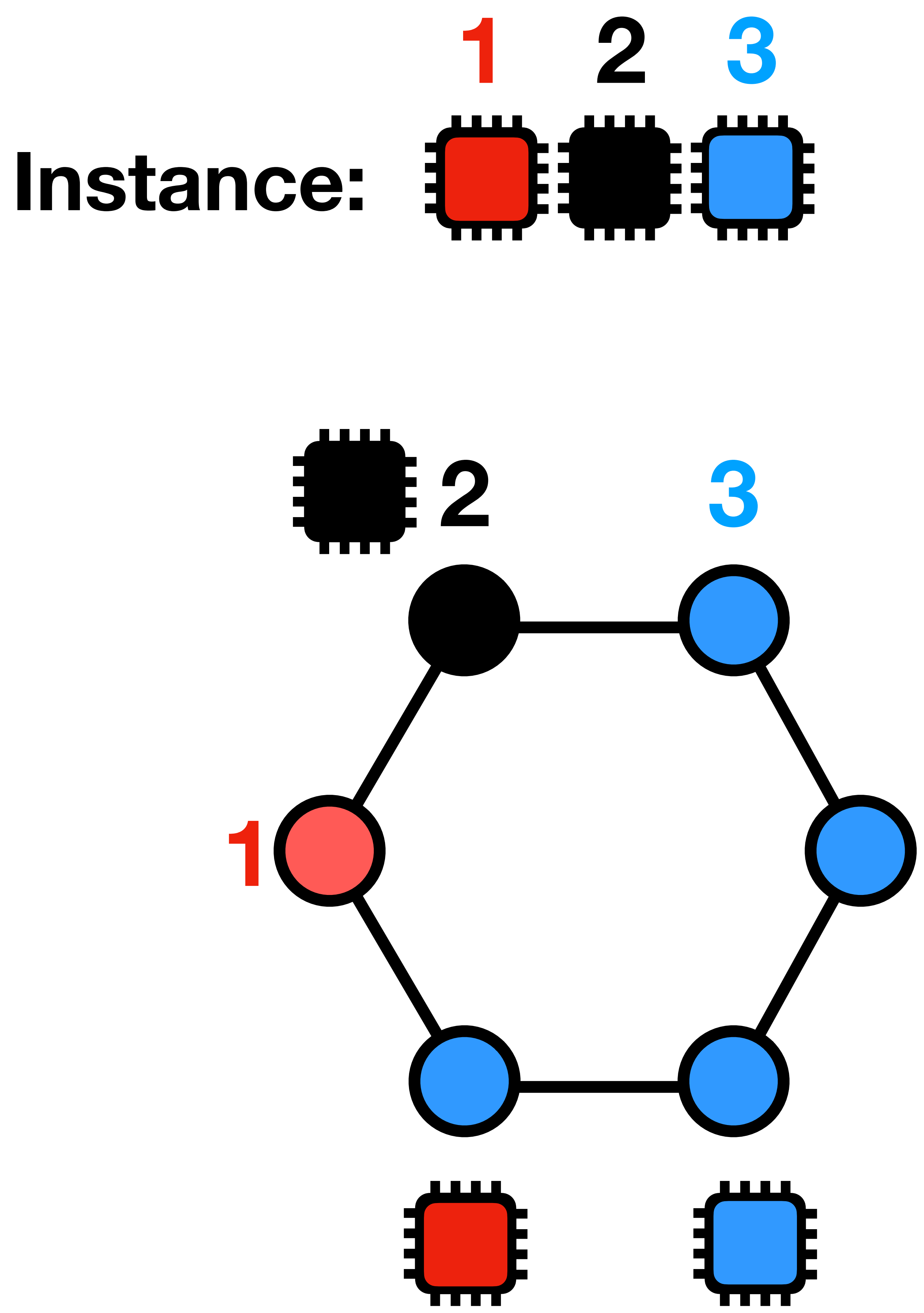


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2. Then **all** processes run the approximate agreement protocol on the cycle.

# A reduction from 2-set agreement

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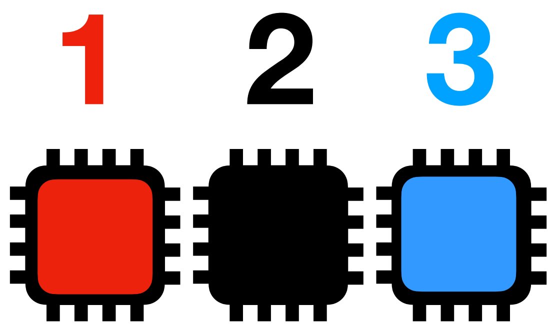


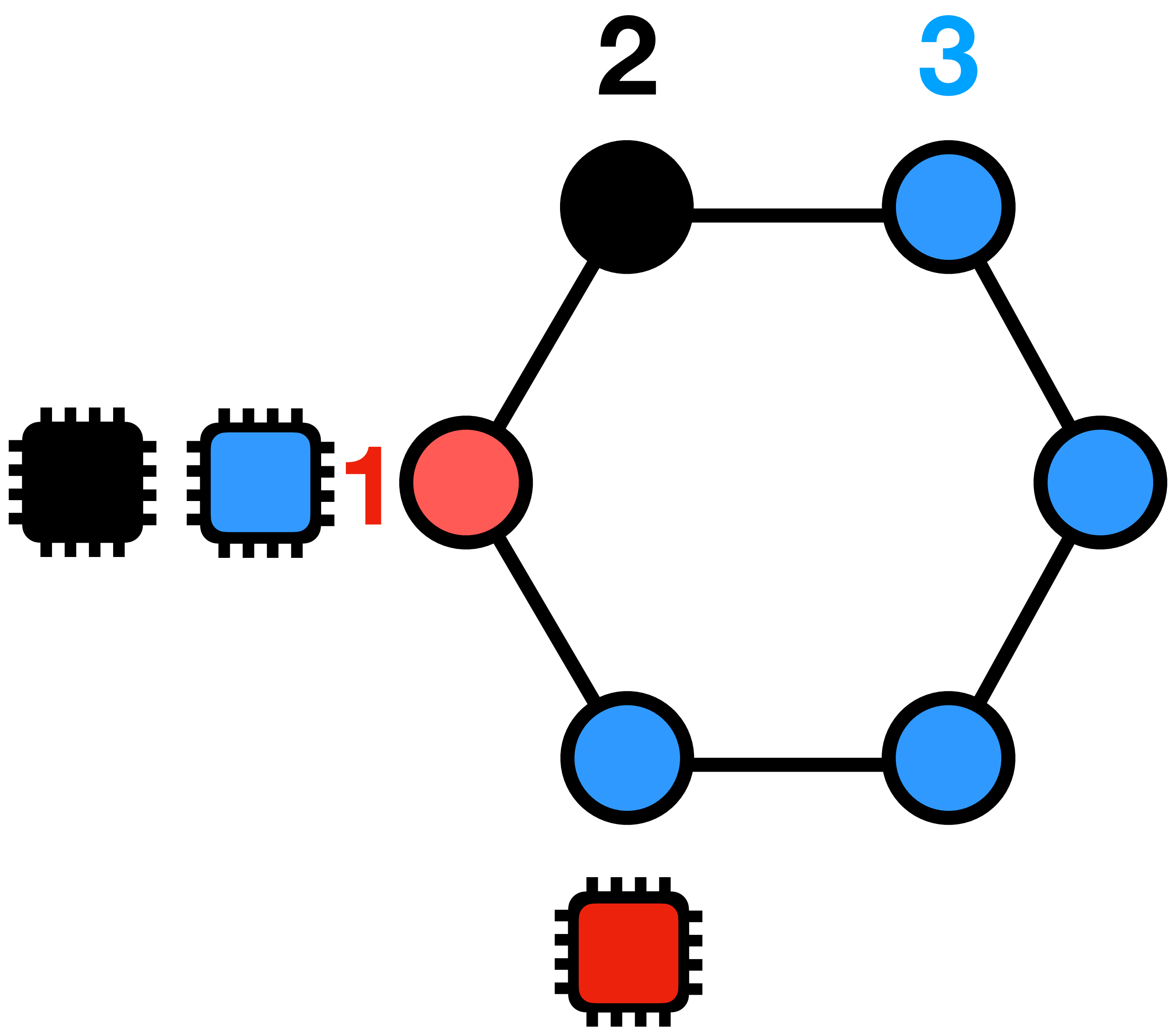


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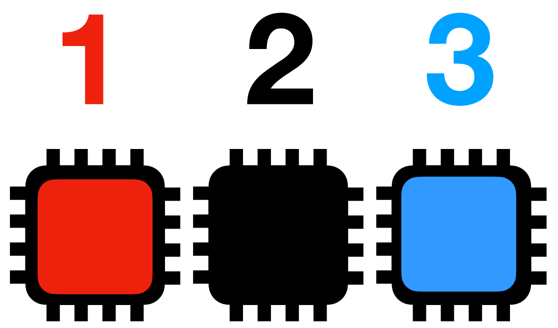
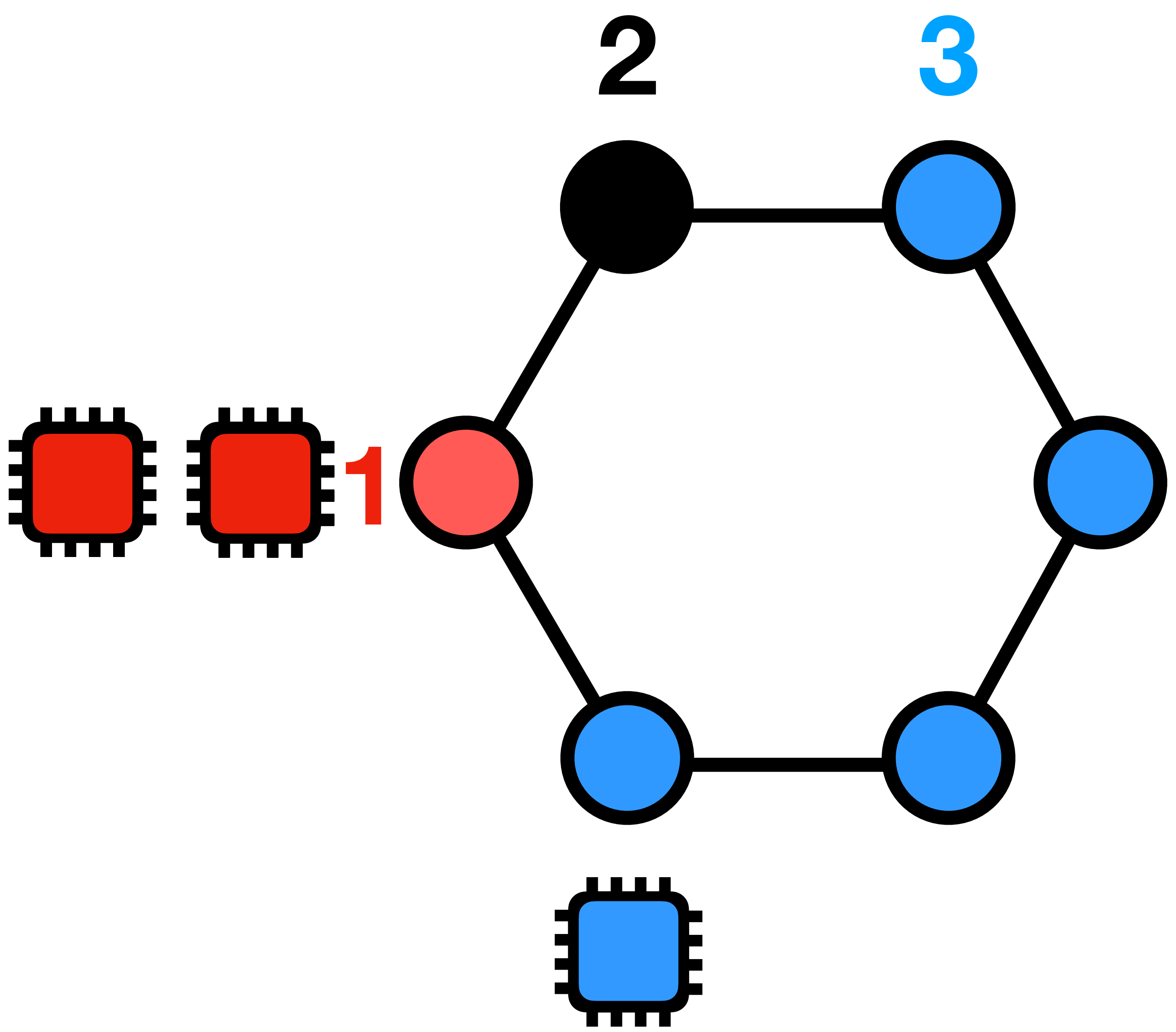
# A reduction from 2-set agreement

Instance: 


1. Processes with inputs **{** 1**,** 3 **}** run a wait-free algorithm for approximate agreement on the **path** obtained by removing the black vertex **2**.

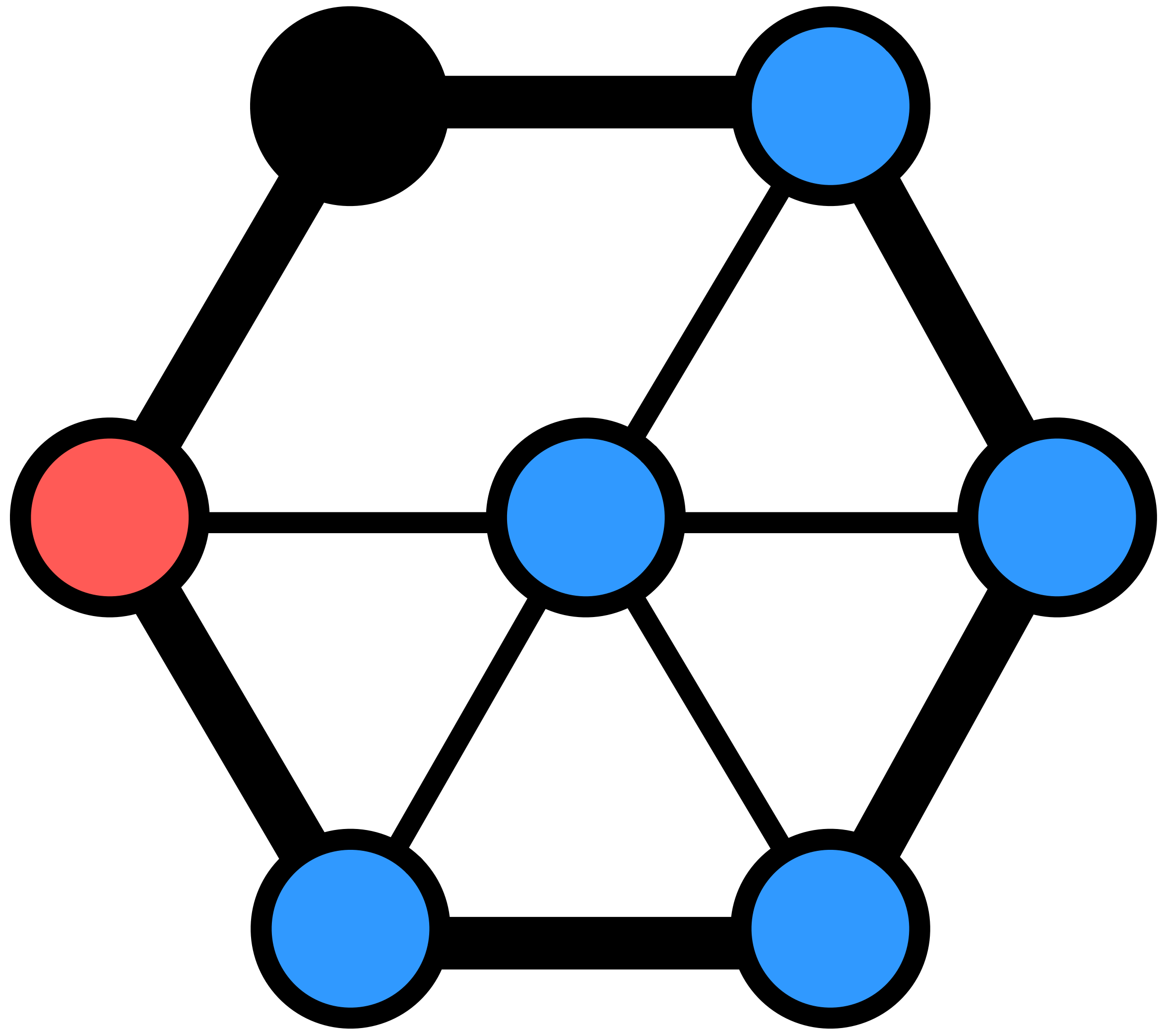
2. Then **all** processes run the approximate agreement protocol on the cycle.

3. Output the ***colour*** of the output vertex.

# **A reduction from 2-set agreement:** beyond cycles

Let $\boldsymbol{L} : \boldsymbol{V} \rightarrow \{\ 1, 2, 3\ \}$ such that

1. there is no triangle with all three colours,
2. there is a cycle ***C*** with three consecutive vertices of colour **1, 2, 3**
3. there is exactly one black node on ***C***



# **A reduction from 2-set agreement:** beyond cycles

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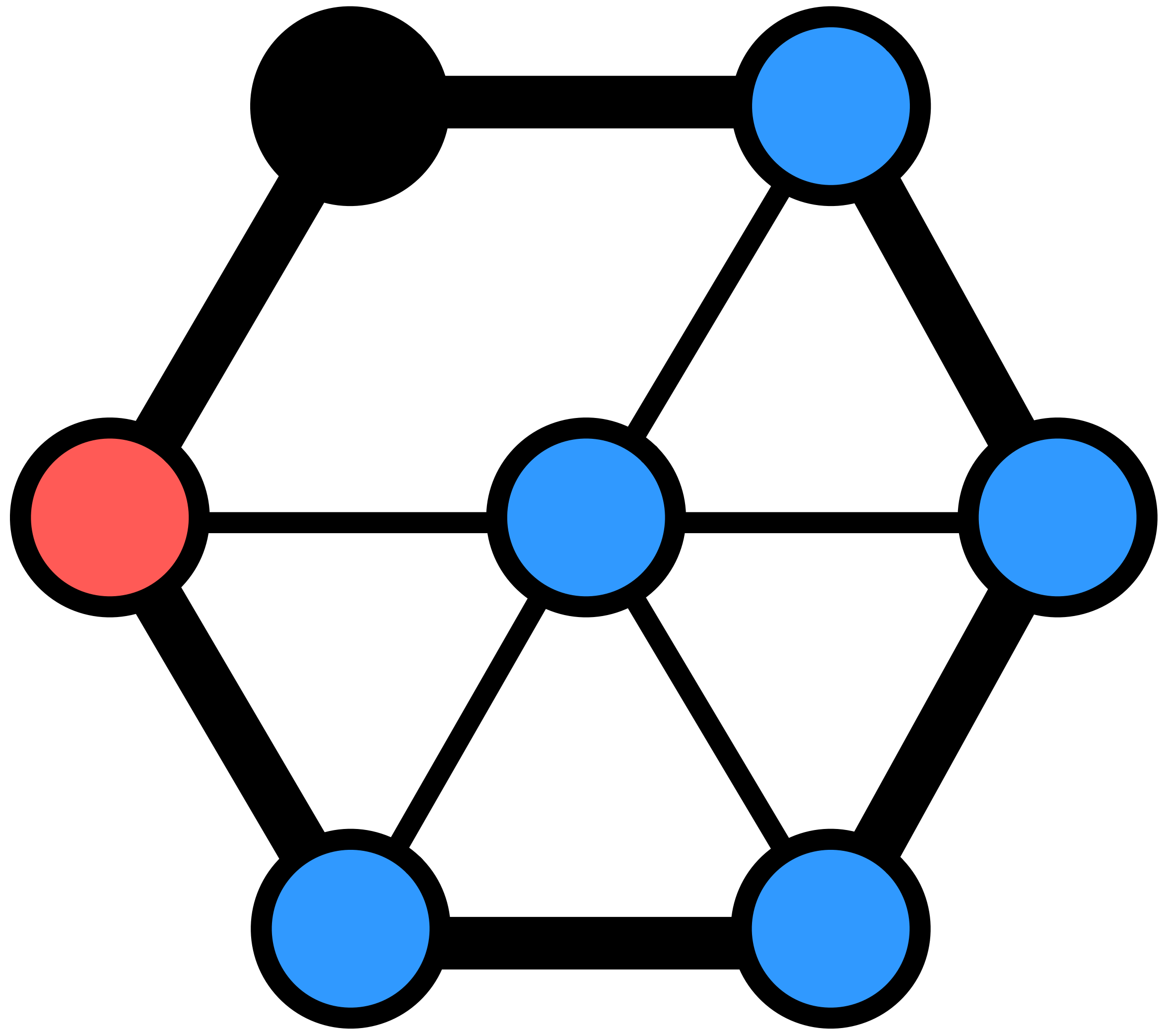
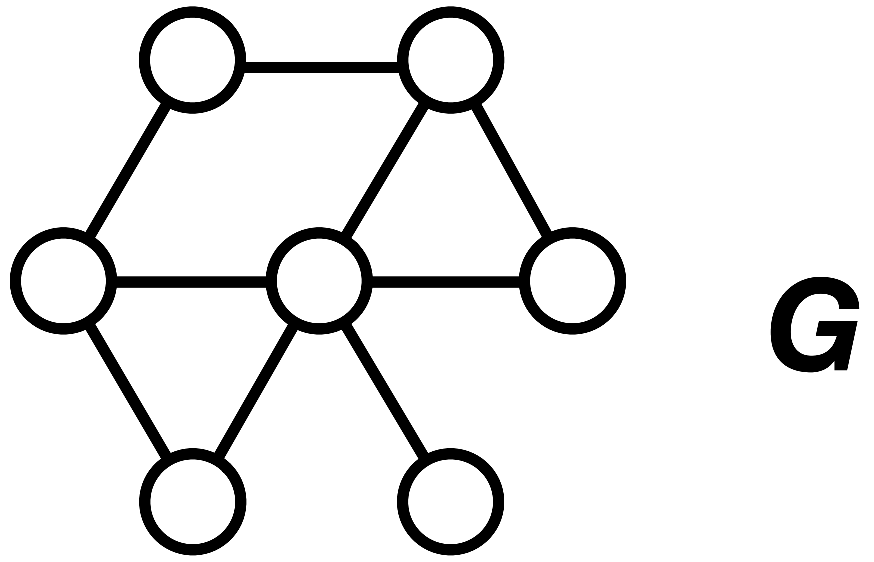
**Theorem (Alistarh, Ellen, Rybicki 2023).**
If $G$ has such an ***impossibility labelling***, then there is no wait-free algorithm for approximate agreement on $G$

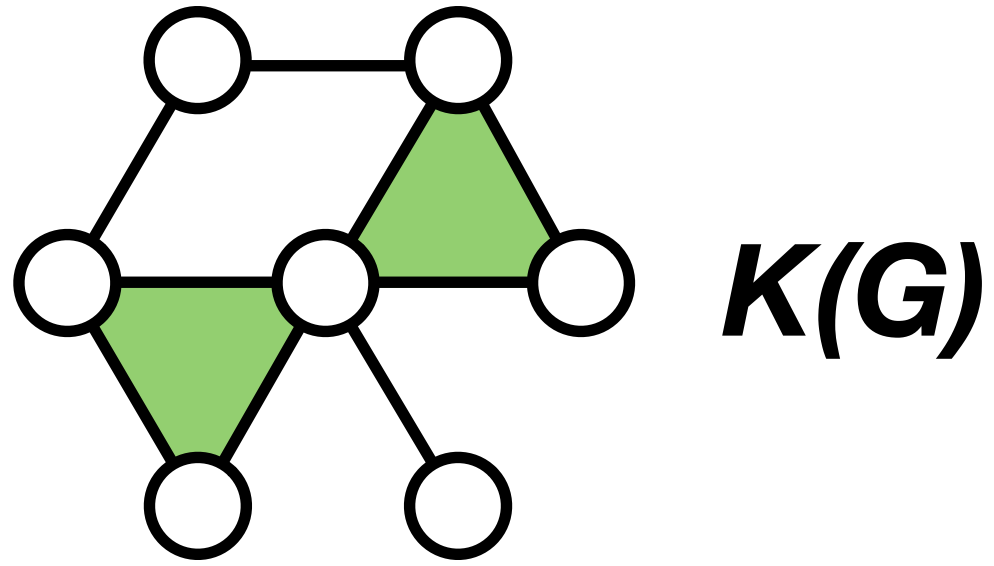
Holds even under **clique validity**.



# Ledent's conjecture

The *complex of cliques of* ***G*** is the complex ***K(G)*** = (***V***, **S**), where **S** is the set of all cliques of ***G*.**

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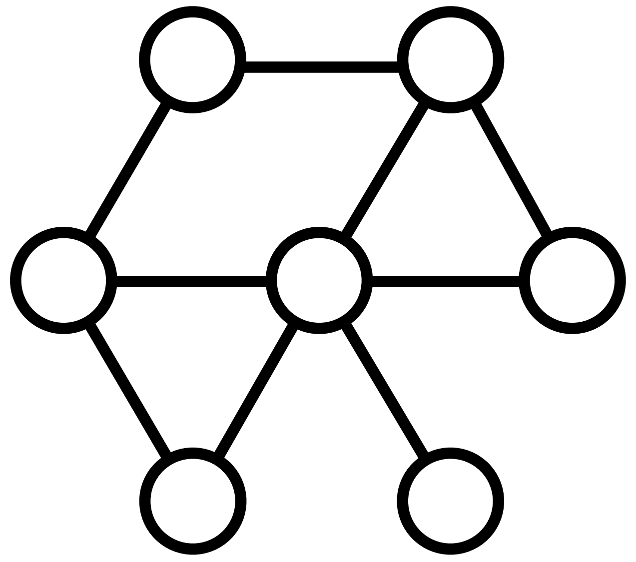


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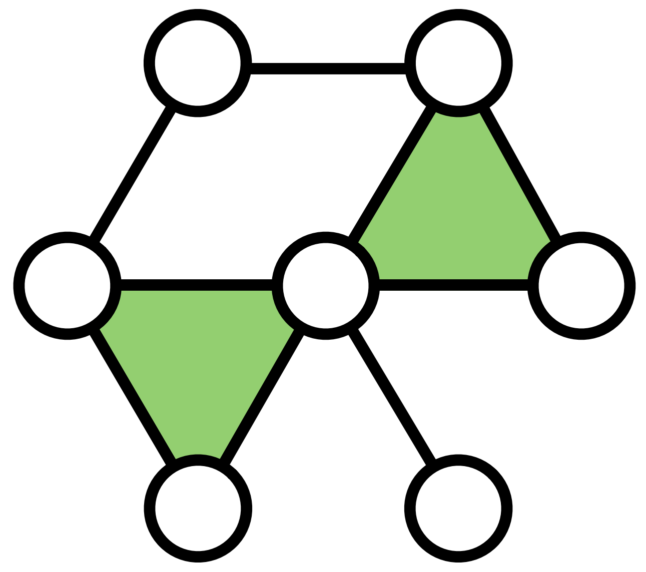
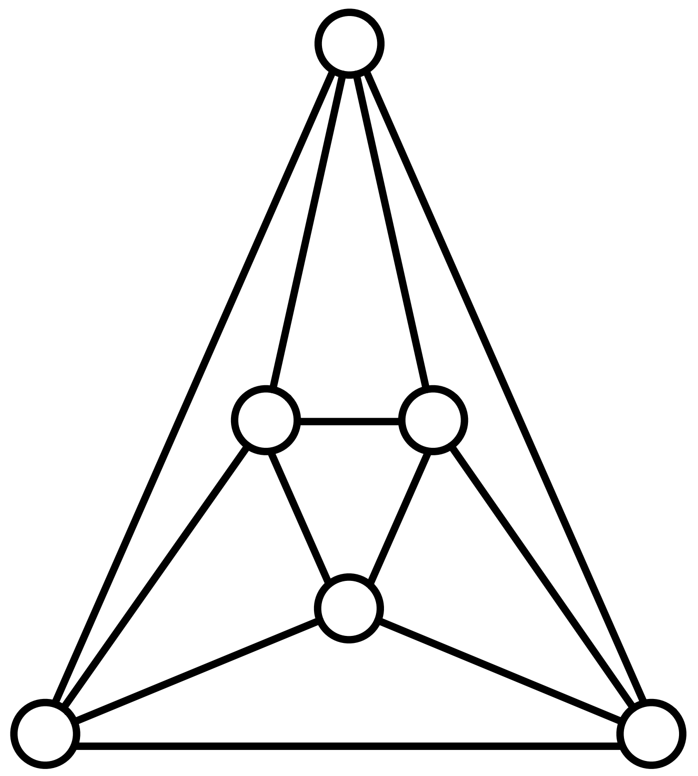


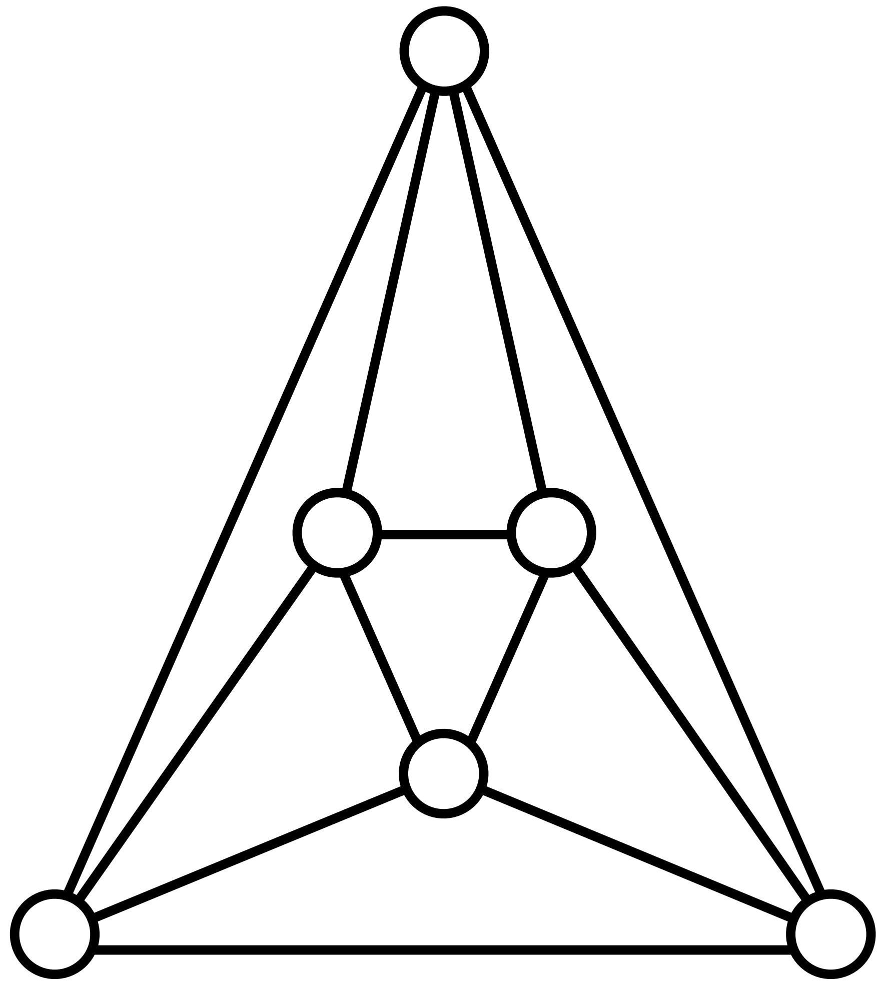

**Interesting case:** *triangulated spheres*
— non-contractible complex of cliques
— no impossibility labelling

Are there wait-free algorithms for such graphs?



octahedron

# Liu’s theorem



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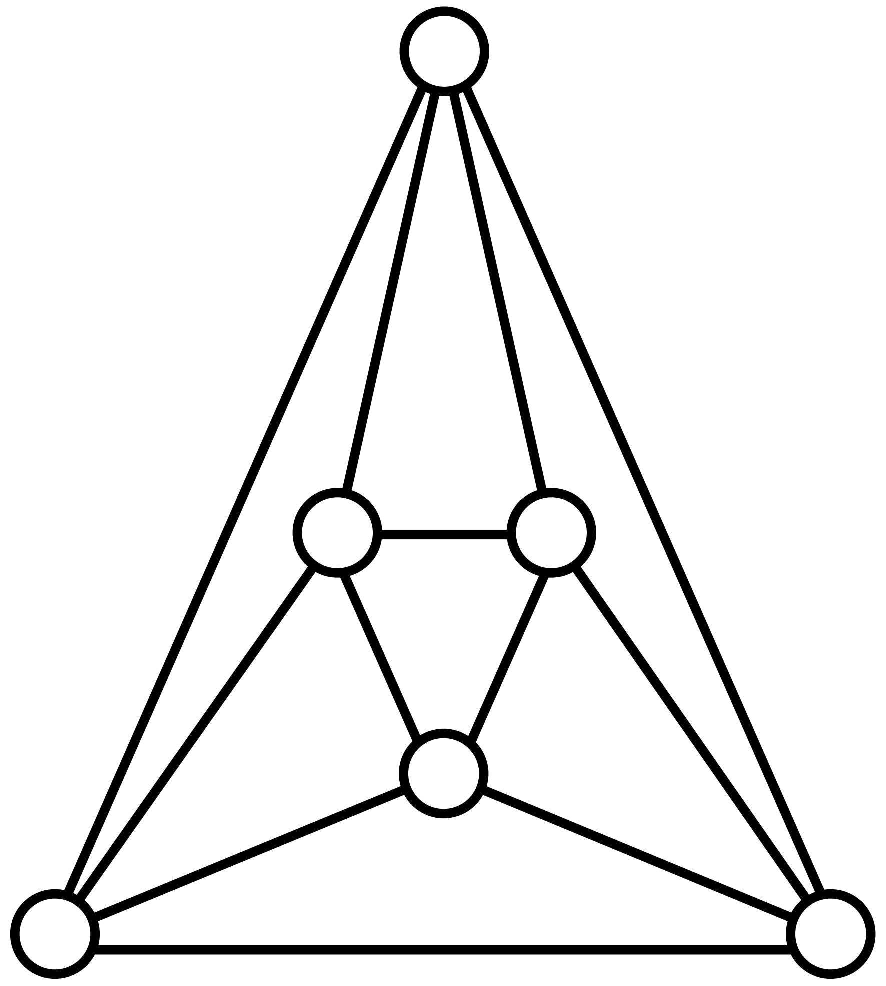

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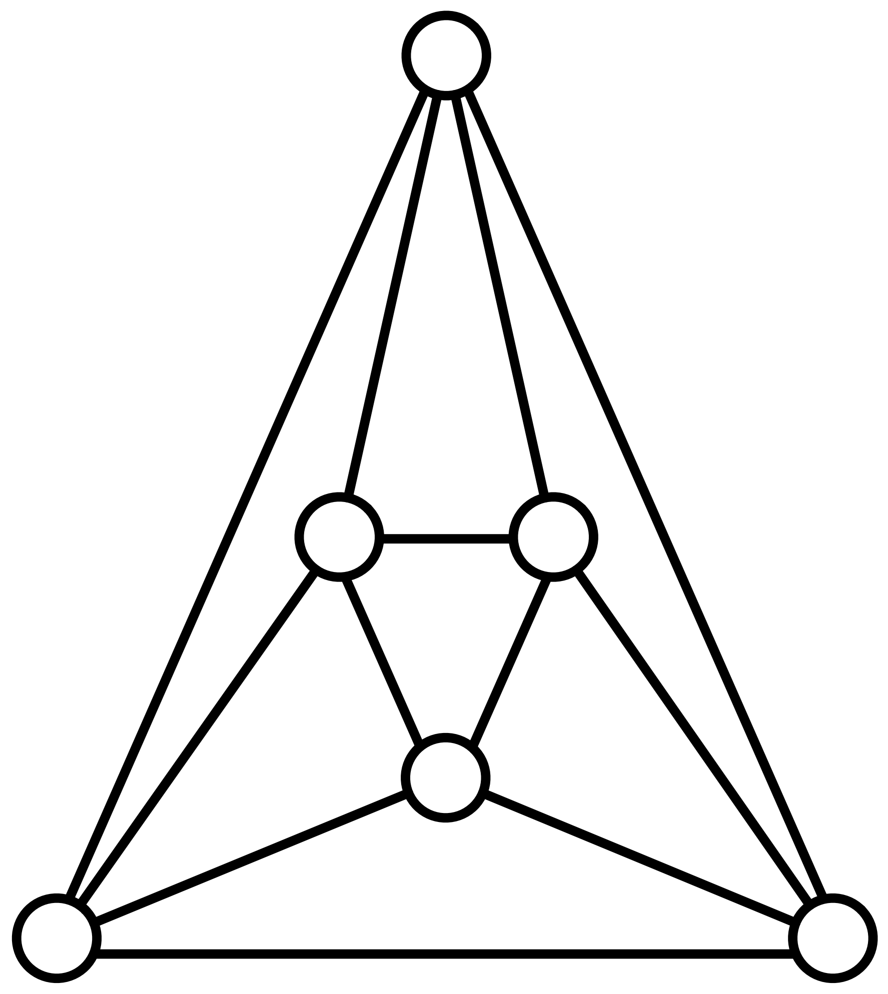
***χ(G):*** chromatic number of ***G***

# Liu's theorem



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For example, **triangulated *d*-dimensional spheres**
satisfy the (*d+1*)-clique containment condition.

**Open problem:** Is there a matching upper bound?

# Summary

## Upper bound techniques

- "iterative **pruning** of convex hull", works in chordal graphs and "nicely bridged" graphs: Alistarh et al. (2023)

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## Lower bound techniques

- **reductions:** Castañeda et al. (2018), Alcántara et al. (2019), Alistarh et al. (2023), Liu (2022)
- **topological proofs:** Alistarh et al. (2023), Liu (2022)

# What else is there?

## Extension-based proofs

- **no “simple” impossibility proofs exist:** e.g., Alistarh et al. (2021), Liu (2022), …

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## Connections to other agreement problems

- **robot gathering:** Castañeda et al. (2018), Alcántara et al. (2019)
- **simplex agreement:** Ledent (2021)
- **multi-valued consensus:** Attiya and Welch (2023)

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**Thank you!**

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