Measurable Combinatorics

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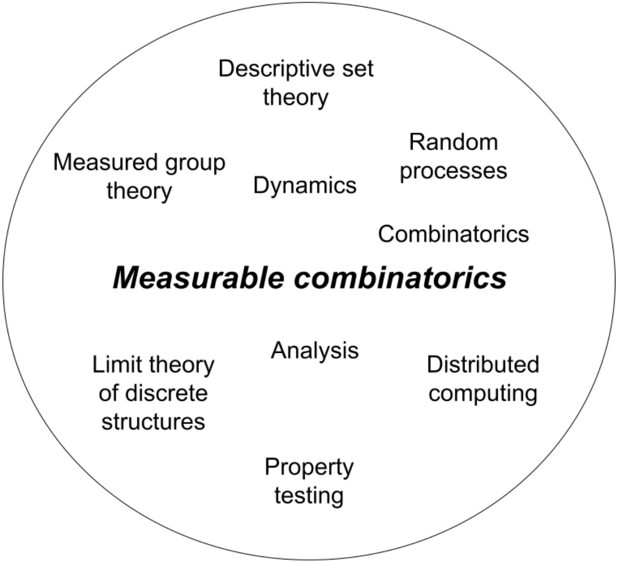
ADGA 2024 @ DISC, 28.10.2024

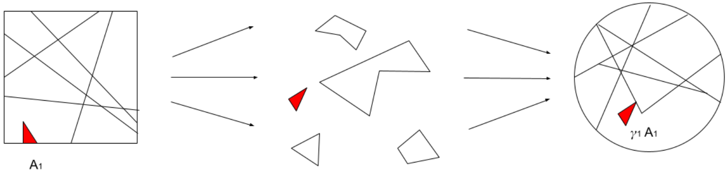


Rosetta stone are *complexity classes* of **Locally Checkable Labeling (LCL) problems**.

- some of the areas can be seen as a model for solving LCL problems,
- *Circle-squaring problem* of Tarski as a motivation for Measurable combinatorics and an illustrative example of the connections

Tarski's circle-squaring problem (1925):

Let $\mathbb{D}$ be a disc of unit area and $\mathbb{S}$ be a square of unit area in $\mathbb{R}^2$. Are $\mathbb{D}$ and $\mathbb{S}$ equidecomposable using rigid motions of $\mathbb{R}^2$?



$$\mathbb{S} = A_1 \sqcup \cdots \sqcup A_k \text{ and } \mathbb{D} = (\gamma_1 \cdot A_1) \sqcup \cdots \sqcup (\gamma_k \cdot A_k),$$

where $\gamma_1, \ldots, \gamma_k$ is a tuple of rigid motions of $\mathbb{R}^2$.

Couple of immediate questions:

What is the motivation behind this problem?

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Original motivation from foundations of measure theory:

- Lebesgue measure (1901) not defined on all subsets of $\mathbb{R}^d$, Vitali sets (1905),
- is there a finitely additive invariant measure defined on all subsets of $\mathbb{R}^d$,
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- if Tarski's problem has *positive answer*, then $\mu(\mathbb{D}) = \mu(\mathbb{S})$ for every finitely additive invariant measure μ defined on *all* subsets of $\mathbb{R}^2$.

Equidecompositions $\leftrightsquigarrow$ Perfect matchings

Recall that our *goal* is to get

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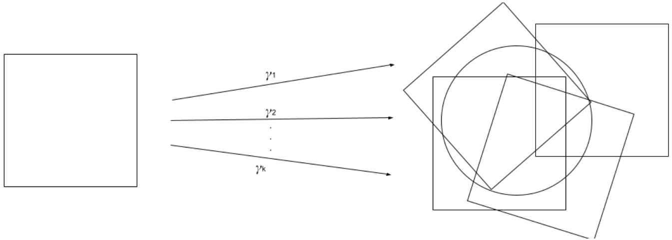
Suppose that someone smart gives us a tuple of rigid motions $(\gamma_1, \ldots, \gamma_k)$ with the promise that they work for the circle-squaring problem.



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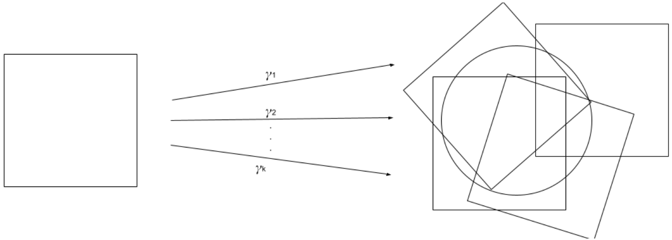

How to find the decompositions? How to interpret this promise?

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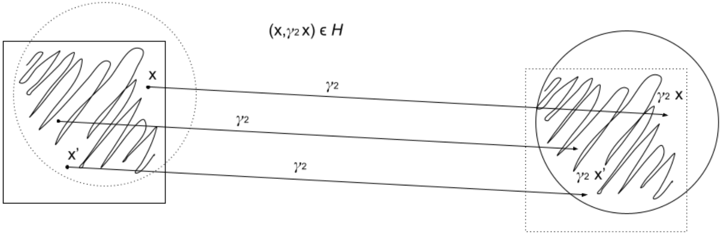

Define a bipartite graph $\mathcal{H}$ with vertex set $\mathbb{D} \sqcup \mathbb{S}$ such that $(x, y) \in \mathbb{D} \times \mathbb{S}$ is an edge if there is $1 \leq i \leq k$ such that $\gamma_i \cdot x = y$.

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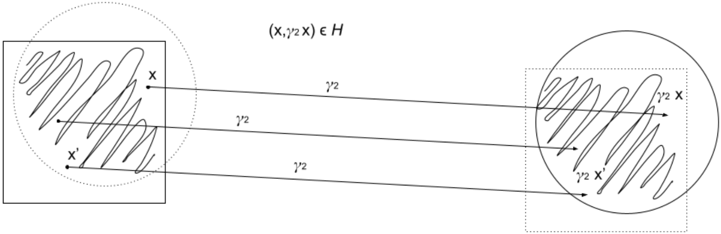

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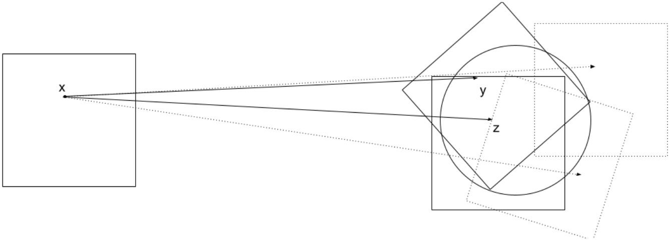

Equidecompositions $\leftrightsquigarrow$ Perfect matchings

Observation. Finding equidecomposition using the tuple $(\gamma_1, \ldots, \gamma_k)$ is equivalent with finding a *perfect matching* in $\mathcal{H}$.

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If $\mathbb{D} = A_1 \sqcup \cdots \sqcup A_k$ is the decomposition, define

$$M = \bigcup_{i=1}^{m} \{(x, \gamma_i \cdot x) \in \mathcal{H} : x \in A_i\}.$$

Trick of Laczkovich

In general, it is hard to control properties of the bipartite graph $\mathcal{H}$ for a given tuple $(\gamma_1, \ldots, \gamma_k)$, in this case, Hall's condition.

Theorem (Laczkovich 1990). $\mathbb{D}$ *and* $\mathbb{S}$ *are equidecomposable using translations only.*

The number of pieces in the proof is around 10^{40}, but people believe it should be around 20.

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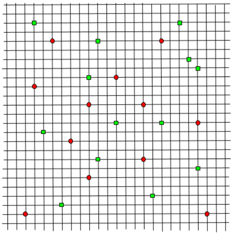
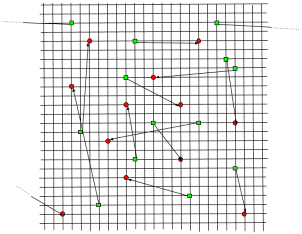
Laczkovich modifies the problem to a special instance of a Locally Checkable Labeling (LCL) problem (with inputs) in $\mathbb{Z}^2$.





Constructible solutions

Algorithmic solutions of the equivalent problem in $\mathbb{Z}^2 \Rightarrow$ *constructible pieces* in the equidecomposition.

Theorem (Grabowski–Máthé–Pikhurko 2017). *$\mathbb{D}$ and $\mathbb{S}$ are equidecomposable with* <u>Lebesgue measurable</u> *pieces using translations only.*

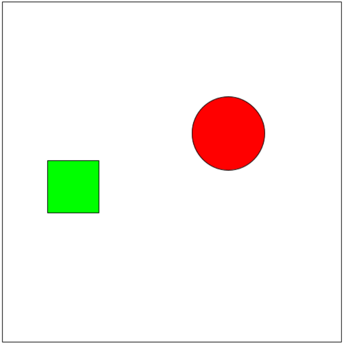
Theorem (Marks–Unger 2017). *$\mathbb{D}$ and $\mathbb{S}$ are equidecomposable with* <u>Borel measurable</u> *pieces using translations only.*

Theorem (Máthé–Noel–Pikhurko 2022). *$\mathbb{D}$ and $\mathbb{S}$ are equidecomposable with* <u>Jordan measurable</u> *pieces using translations only.*

The number of pieces in the proof is around 10^{200}, but people believe it should be around 20.

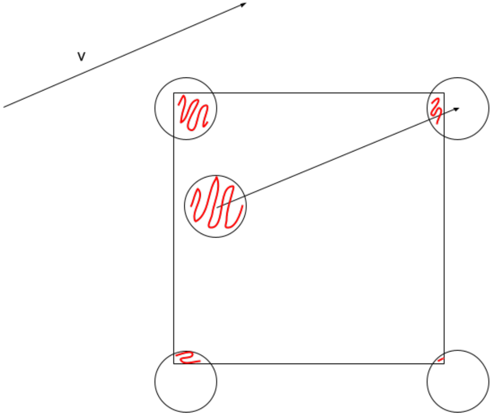
Trick of Laczkovich

Let $[0, \alpha) \times [0, \alpha)$ be a large enough torus that contain disjoint fixed copies of $\mathbb{D}$ and $\mathbb{S}$.



Trick of Laczkovich

Translation of $[0, \alpha) \times [0, \alpha)$ by a vector $v \in \mathbb{R}^2$ works as in the SNAKE game.

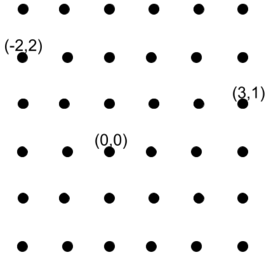


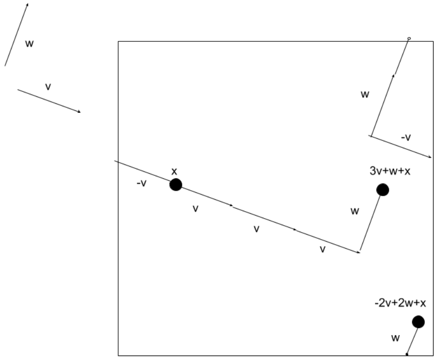
Localizing the problem

Fix two random translations v and w, and consider any $x \in [0, \alpha) \times [0, \alpha)$.

There is a correspondence between $\mathbb{Z}^2$ and the orbit

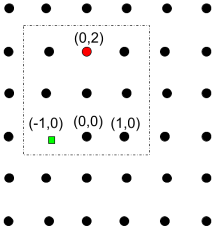
$$\{n \cdot v + m \cdot w + x : (n, m) \in \mathbb{Z}^2\}.$$

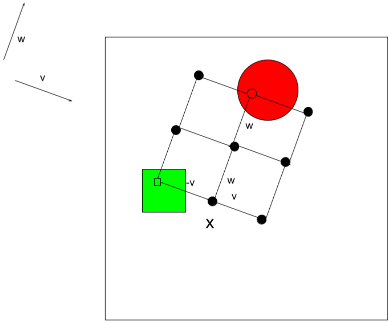




Localizing the problem

Moreover, we can mark when we hit $\mathbb{D}$ and $\mathbb{S}$.

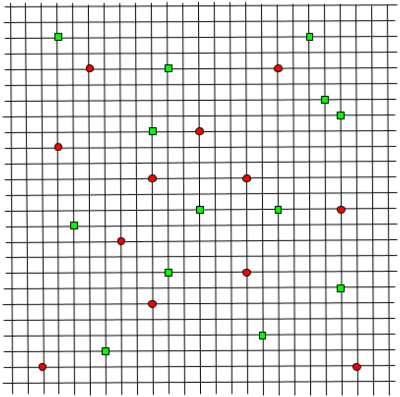




$\rightsquigarrow \mathbb{Z}^2$ *with input labels.*

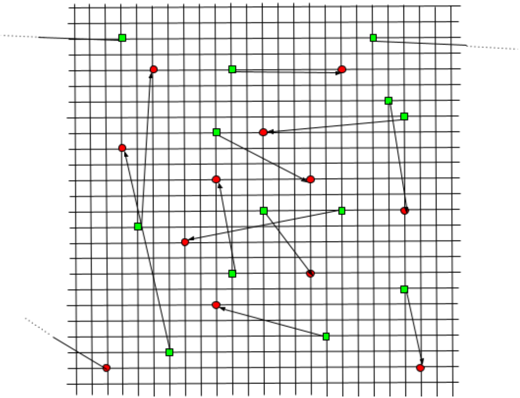
Localizing the problem

Theorem (Laczkovich). For every x, the inputs of $\mathbb{D}$ and $\mathbb{S}$ are equidisitrbuted in $\mathbb{Z}^2$.



Localizing the problem

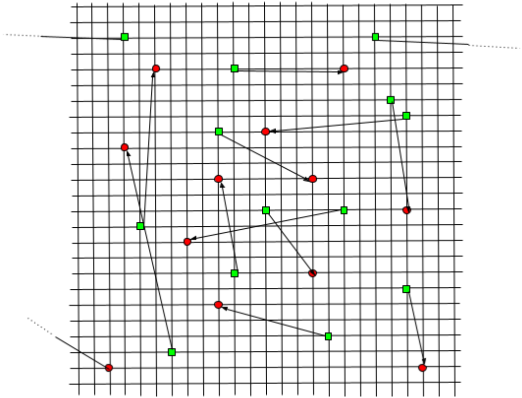
To solve Tarski's problem it is enough to find a pairing between these inputs of uniformly bounded distance in the graph, say $m \in \mathbb{N}$.



Then taking all the elements of the form $\gamma = av + bw$, where $|a|, |b| \leq m$ is the desired "smart" tuple of rigid motions.

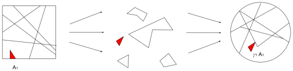
Localizing the problem

Producing a pairing between these inputs by an efficient *local algorithm* impacts the definability/measurability of the final pieces.

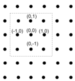


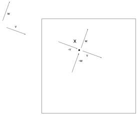
Tarski's problem

- problem in measure theory and geometry,

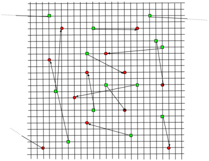


- dynamical interpretation using an action of $\mathbb{Z}^2$ on a torus





- localization to LCL problem on orbits



- local algorithms solving this problem $\rightsquigarrow$ measurable solutions.

Measurable combinatorics

Systematic study of combinatorial questions on graphs that come, and are motivated by problems, from dynamics using *localization* and *local algorithms/rules*.

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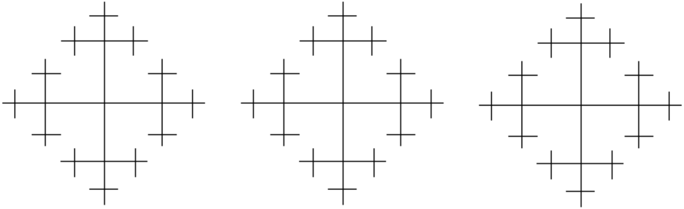
For example, measurable actions of free group on two generators, $\mathbb{F}_2 \curvearrowright (X, \mu)$



correspond to combinatorics on regular trees.

Connections

Theorem (Bernshteyn 2023). *Efficient local algorithms give automatically the existence of measurable solutions.*

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- (Bernshteyn) Vizing's edge coloring based on augmenting chain technique of G.–Pikhurko developed for measurable edge colorings,
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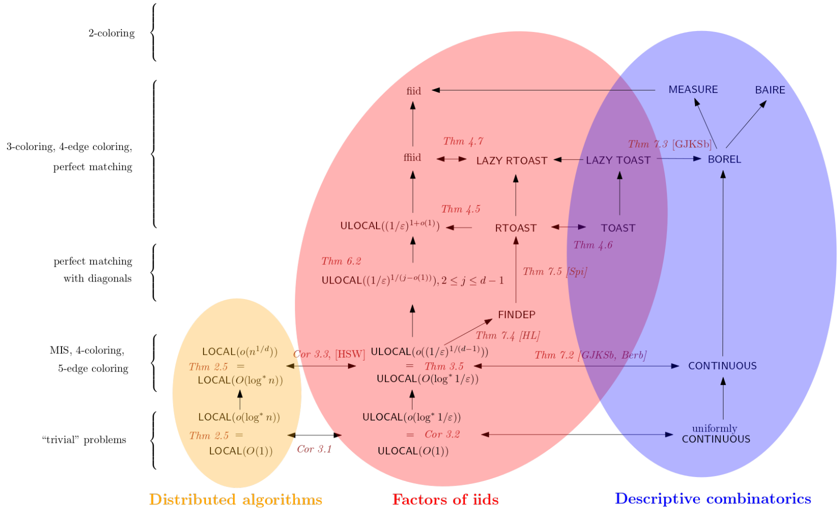
Theorem *Sometimes the setups are completely equivalent.*

- (Bernshteyn, Brandt–Chang–Grunau-G.–Rozhoň–Vidnyánszky) $O(\log^*(n))$ regime in the deterministic LOCAL model and the existence of continuous colorings,
- (Brandt–Chang–Grunau-G.–Rozhoň–Vidnyánszky) $O(\log(n))$ regime in the deterministic LOCAL model for regular trees and the existence of Baire measurable colorings.

Rosetta stone on grid graphs (G.-Rozhoň)



Finitary factor of iid model of distributed computing (Holroyd–Schramm–Wilson)

A version of a randomized LOCAL model, where the algorithm does not have access to the size of the graph.

Number of rounds at each vertex is a random variable that is almost surely finite. Complexity measured by the tail decay of this variable.

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<u>**Open problem:**</u> Study the complexity hierarchy of LCL problems in this setup.

Happy to explain more during the week.

Thank you!