Recent Advances in Deterministic Distributed Algorithms

Christoph Grunau (ETH Zurich)

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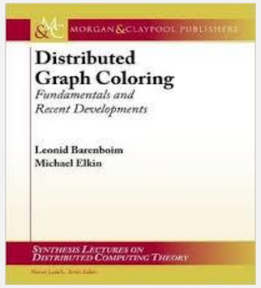

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8 out of the 17
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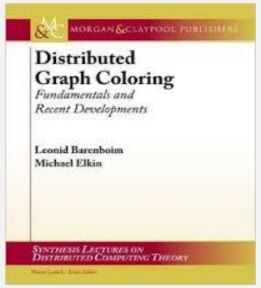

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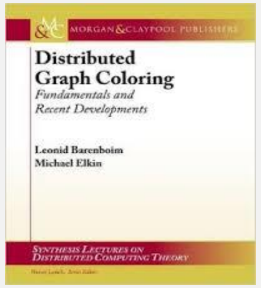

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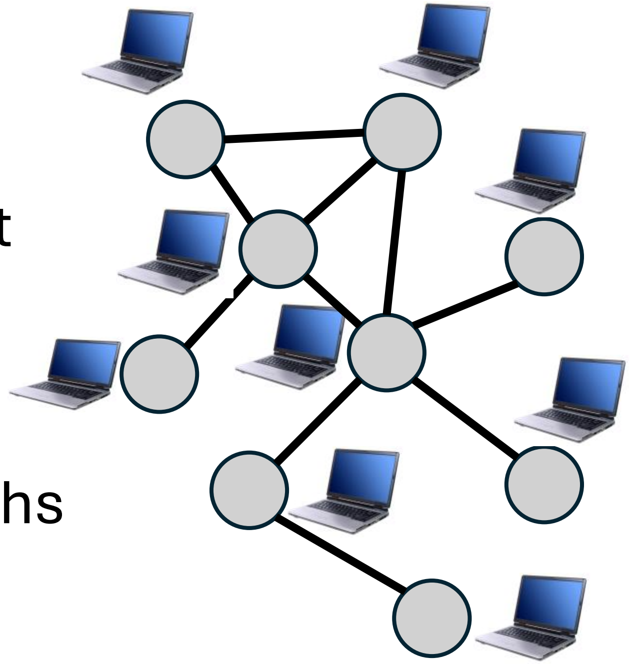
This talk: Present some of the key techniques used in the SOTA algorithms

The LOCAL Model of Distributed Computing *[Linial'87]*

- Undirected graph G = (V,E)
  - V = Processors with unique IDs from $\{1, \ldots, poly(n)\}$
  - E = Communication link

- Synchronous rounds
  - **Unbounded** local computation
  - **Unbounded** message sizes

- After **r** founds: each node needs to know its part of the output
  r = O(diameter) is always possible



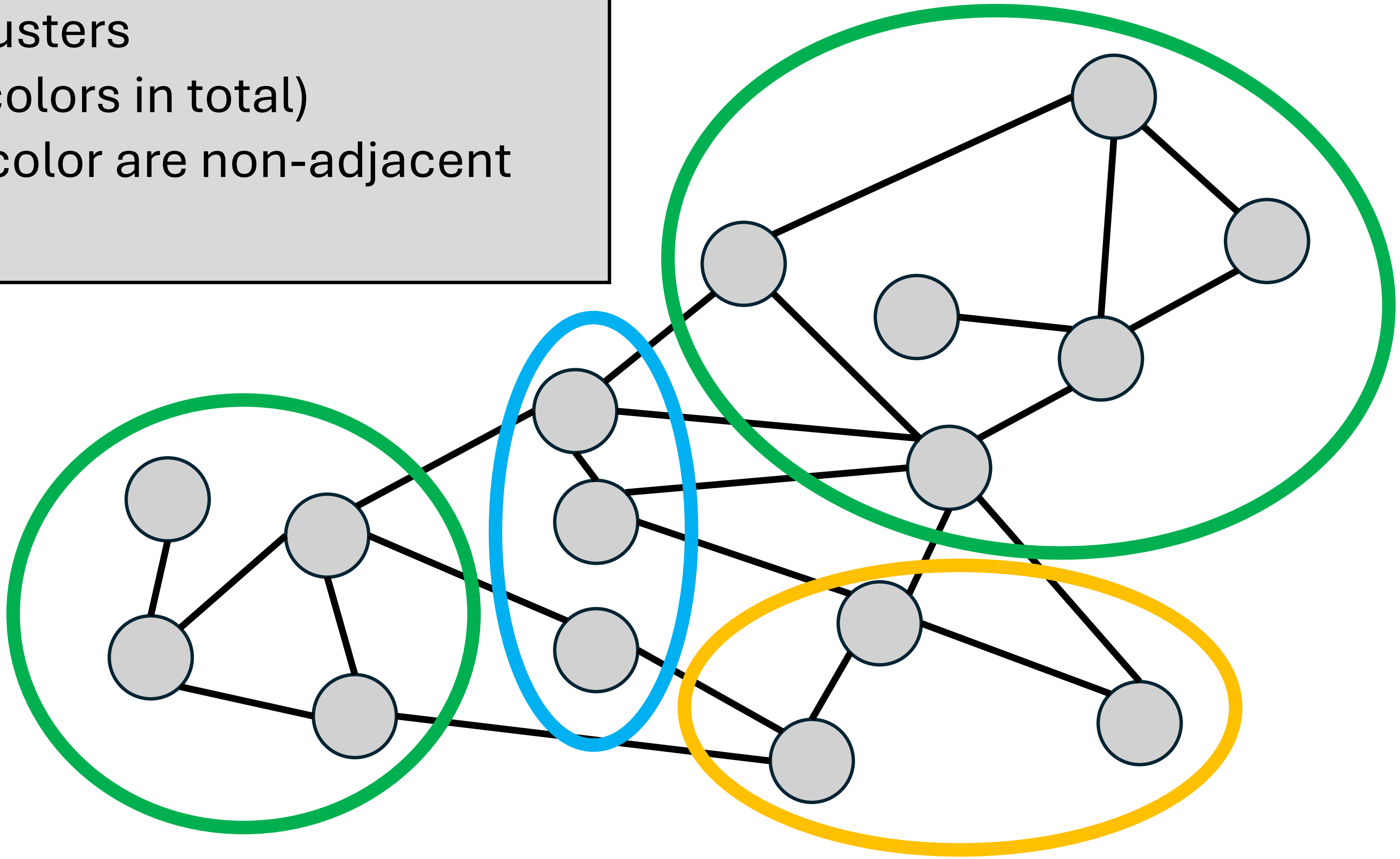
Generic approach: Break the graph into small-diameter subgraphs

Reminder: Network Decomposition & MIS

(C,D) Network Decomposition

- Graph is partitioned into clusters
- Each cluster is colored (**C** colors in total)
  - Clusters with identical color are non-adjacent
- Cluster diameter at most **D**

(3,3) Network Decomposition



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Claim:

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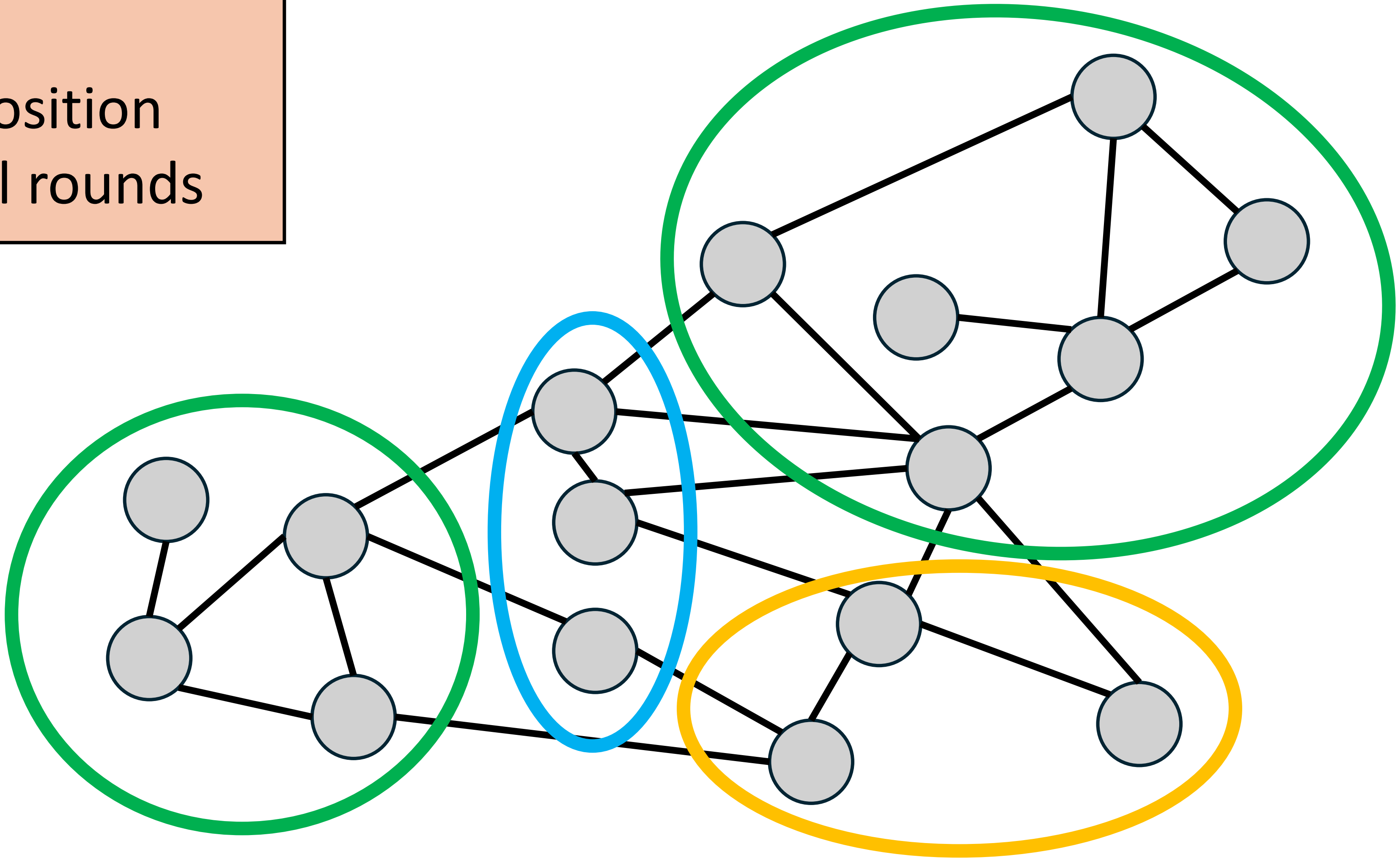
⇨ MIS in O(CD) additional rounds

(3,3) Network Decomposition



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Existential: can get C & D = O(log n). Implicit in [Awerbuch, Peleg '89]

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[Ghaffari, **G**'24] : Deterministic (O(log n), O(log n))-ND in $\tilde{O}(\log^2 n)$ rounds.

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Remarks:

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Corollary 1: Deterministic Maximal Matching and $(\Delta + 1)$-coloring in $\tilde{O}(\log^{5/3} n)$ rounds.

Corollary 2: Randomized $(\Delta + 1)$-coloring in $\tilde{O}(\log^{\frac{5}{3}} log\ n)$ rounds.

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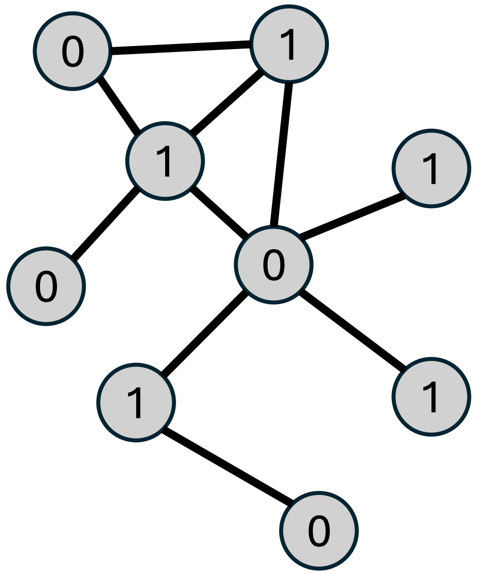
Let's start with a toy problem: Max Cut

Toy Problem: Max-Cut

- Let $X_v \in \{0, 1\}$ indicate the side of the cut for node v.



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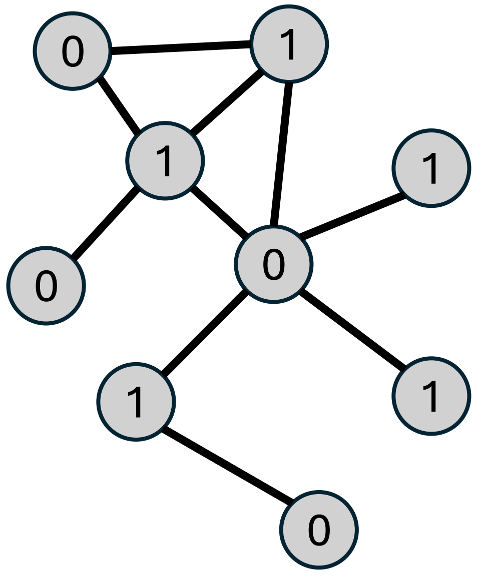

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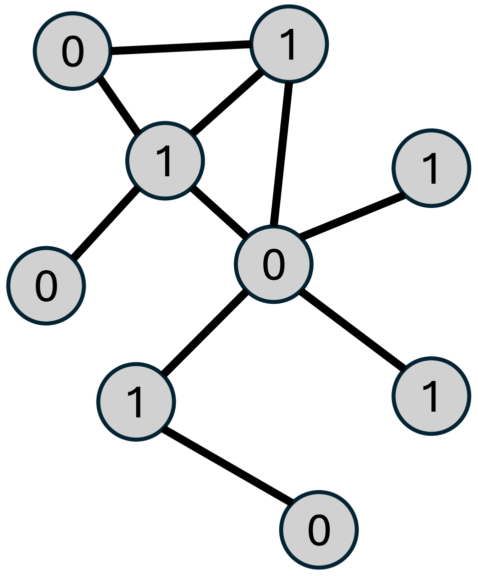

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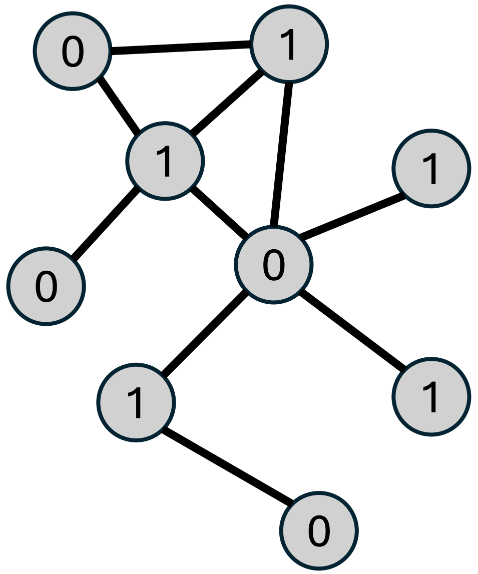

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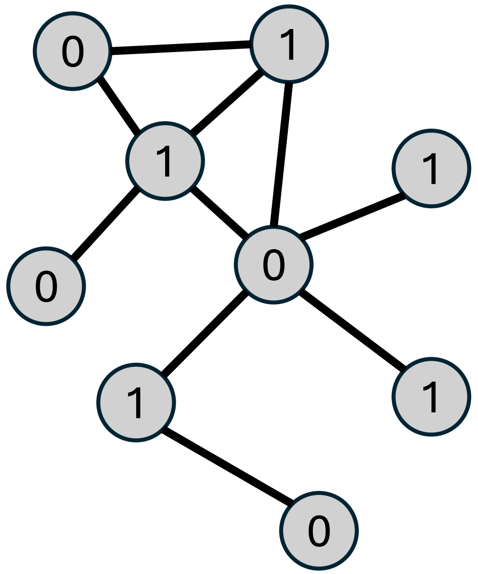

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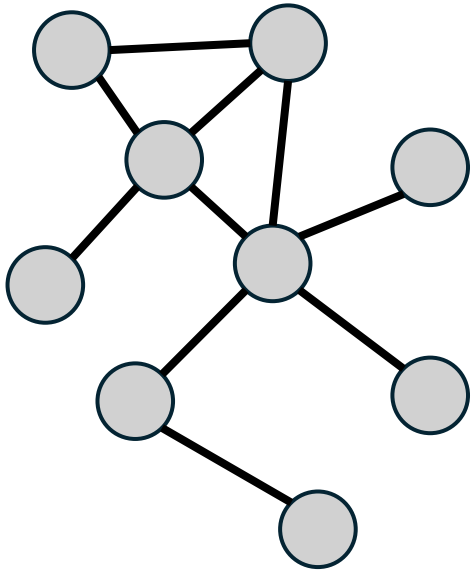
Issue: Highly Sequential

Determinstic Distributed $(\frac{1}{2} - \varepsilon)$-Approx.

Algorithm:
1.) Compute a coloring such that at most an ε-fraction of edges are monochromatic
2.) Remove all monochromatic edges.
3.) Iterate through the colors, fixing all nodes of the same color at once



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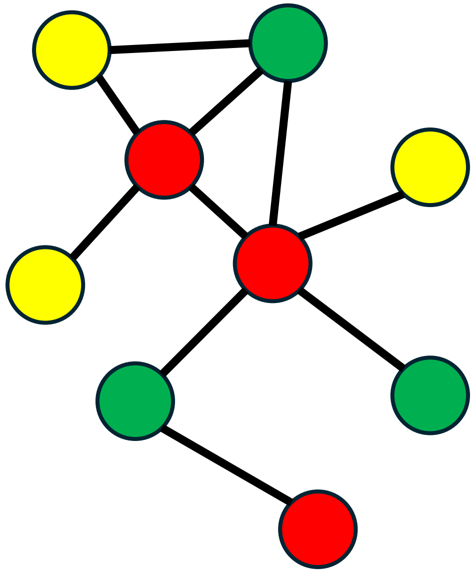

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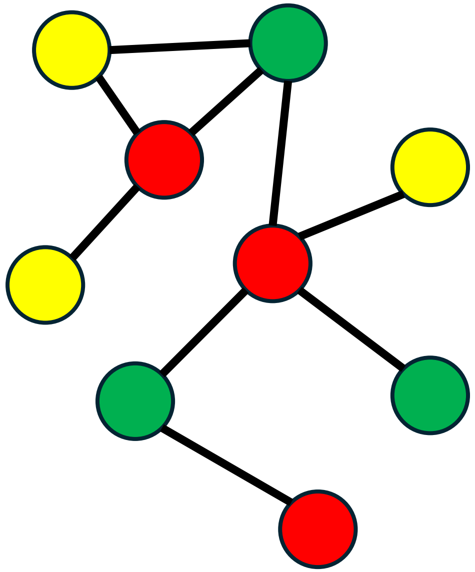

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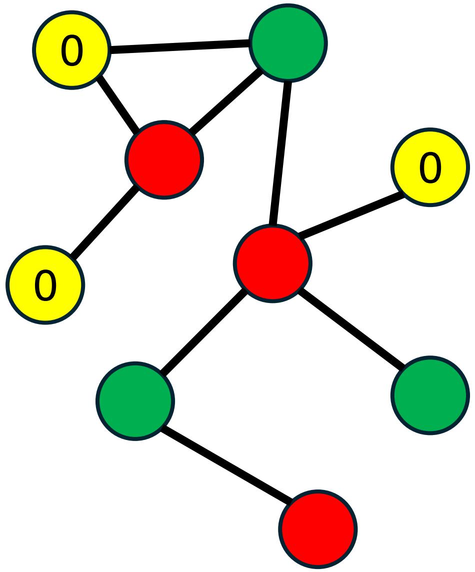

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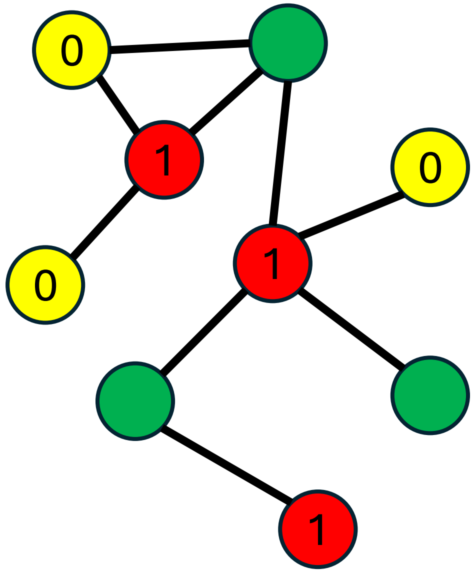

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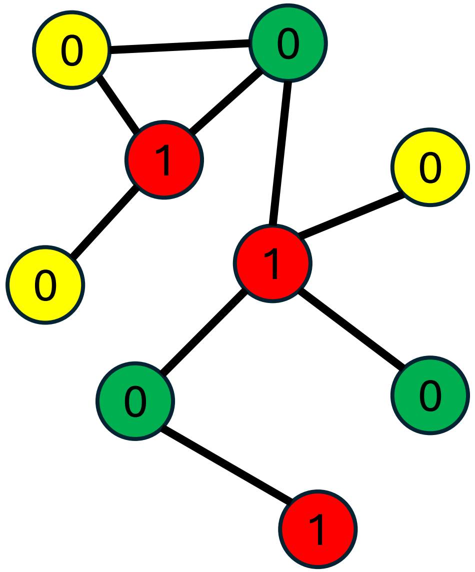

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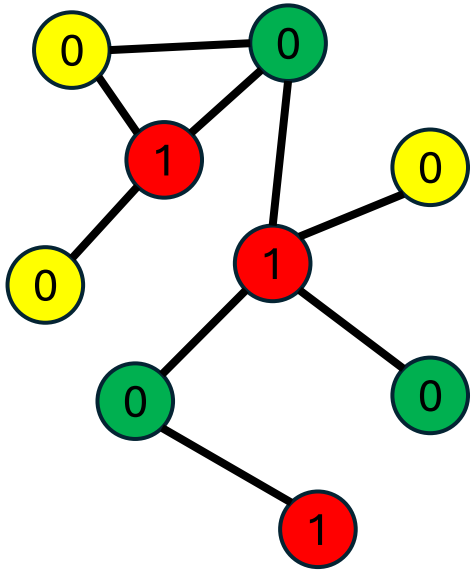
[Ghaffari, Kuhn' 21]:
Such a coloring with O(1/ ε) colors can be computed in O(1/ ε+$\log^* n$) rounds



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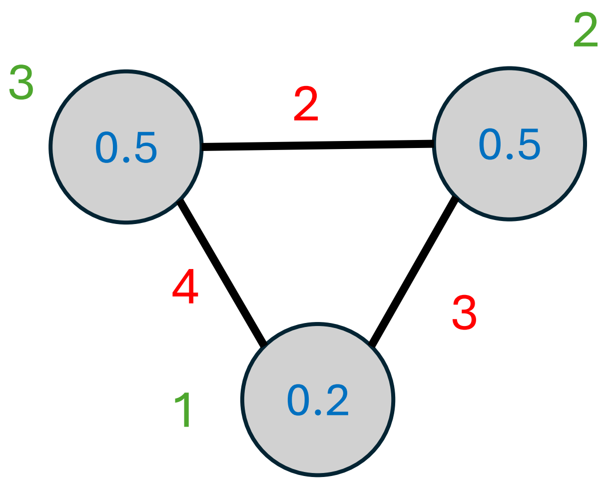

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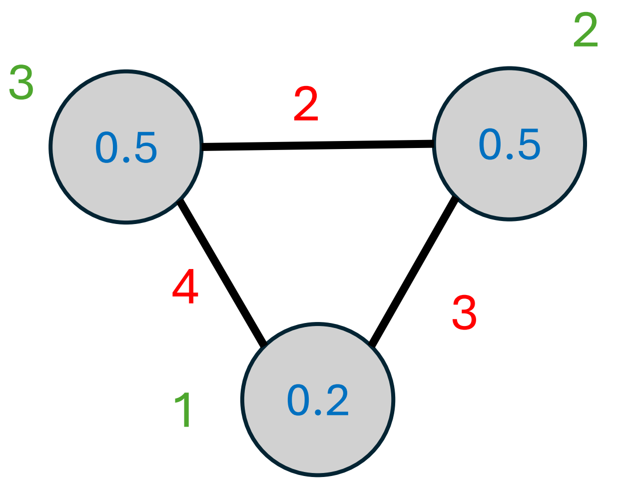
Notation:

$$\Phi(\vec{x}) := \sum_{v\in V} ut(v)x(v) - \sum_{\{u,v\}\in E} cost(uv)x(u)x(v) \; (\vec{x} \in [0,1]^V)$$

"Expected utility" – "Expected cost"



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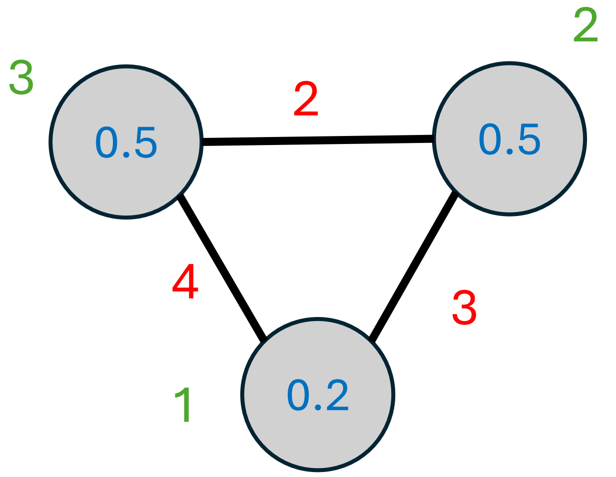

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Theorem [Faour, Ghaffari, G, Kuhn, Rozhoň'23]

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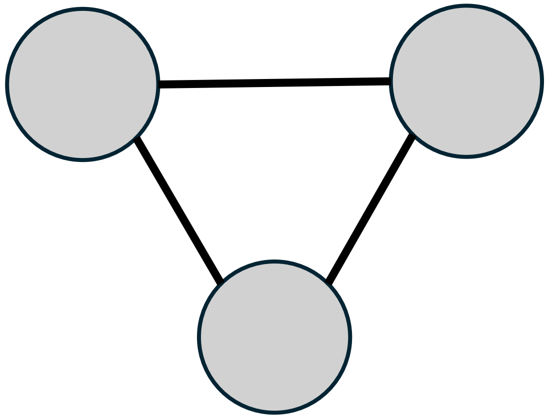

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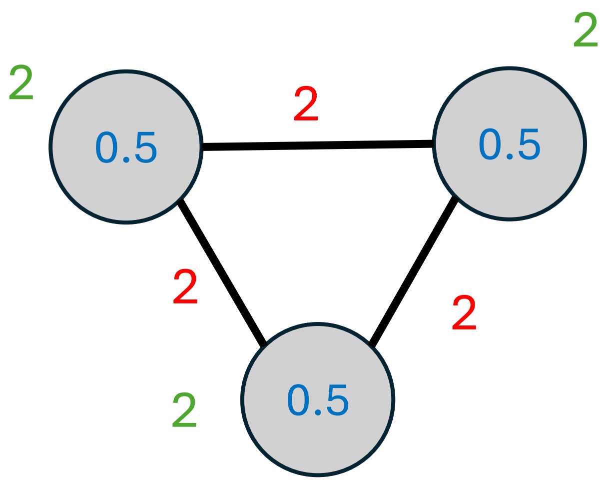

Max Cut:

- $p(v)$= 0.5
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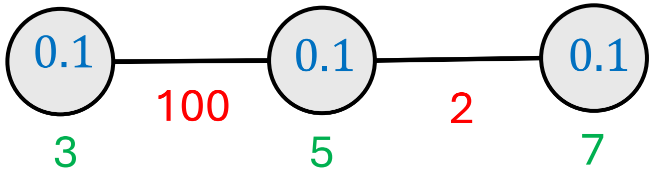

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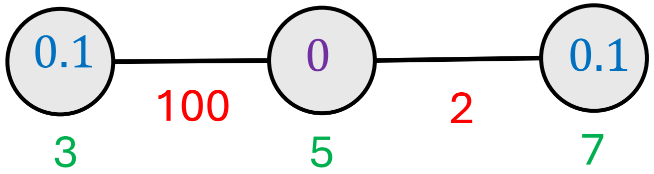

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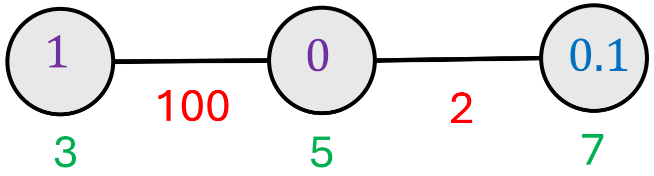

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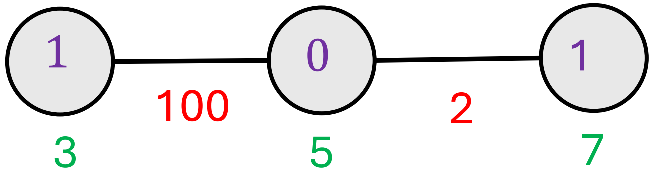

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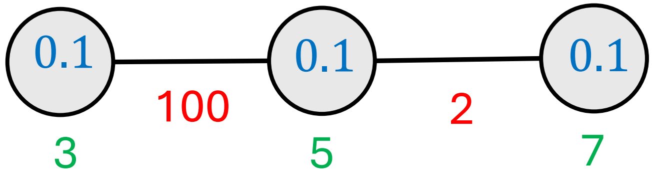

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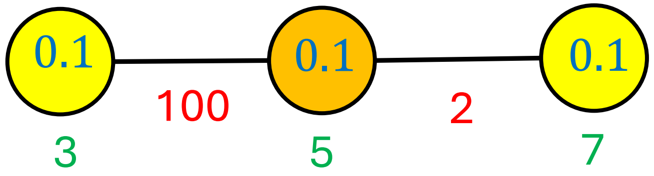

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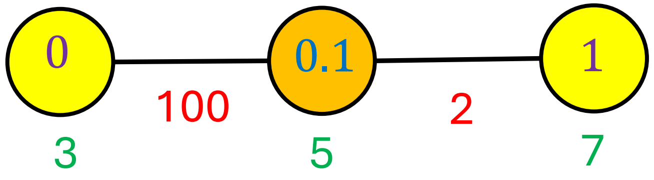

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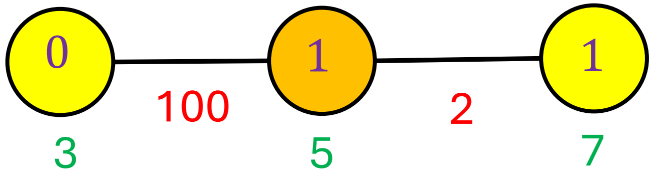

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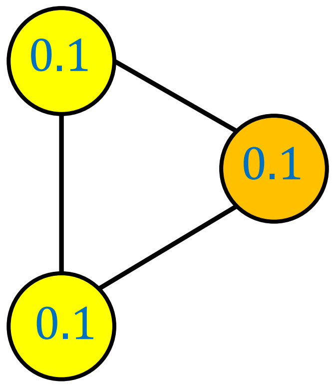

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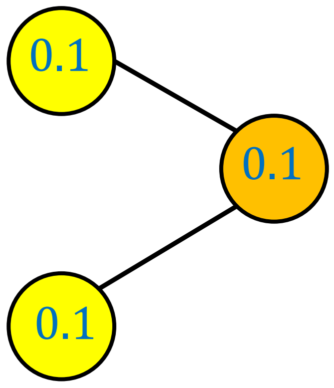

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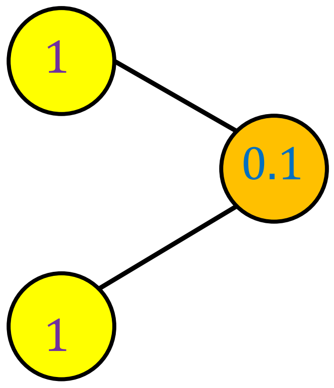

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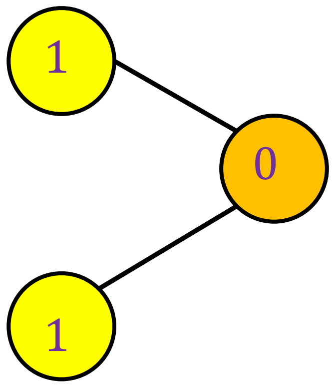

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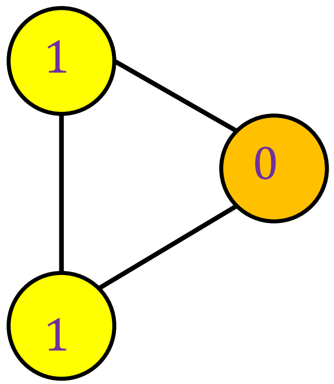

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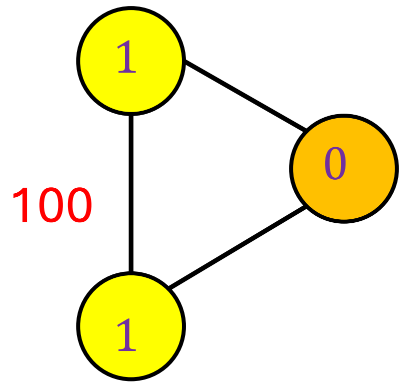

Before: $100 \cdot 0.1 \cdot 0.1$
After: $100 \cdot 1 \cdot 1$

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Given a $\frac{1}{2^i}$-integral $\overrightarrow{p_i}$, one can compute a $\frac{1}{2^{i-1}}$ -integral $\overrightarrow{p_{i-1}}$ with $\Phi(\overrightarrow{p_{i-1}}) \geq \left(1-\frac{1}{10I}\right)\Phi(\overrightarrow{p_i})$ in $O(I)$ rounds

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- Overall round complexity: $O(I^2)$

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1.) Compute a coloring with $O(I)$ colors so that monochromatic edges contribute $\leq \frac{1}{40I}\Phi(\overrightarrow{p_i})$.

Formally: $\sum_{\{u,v\}\in E^{mono}} cost(uv)(\overrightarrow{p_i})_u(\overrightarrow{p_i})_v \leq \frac{1}{40I}\Phi(\overrightarrow{p_i})$

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Potential Analysis:

$$\begin{aligned}
\Phi(\overrightarrow{p_{i-1}}) &= \Phi_{E\setminus E^{mono}}(\overrightarrow{p_{i-1}}) - \sum_{\{u,v\}\in E^{mono}} cost(uv)(\overrightarrow{p_{i-1}})_u(\overrightarrow{p_{i-1}})_v \\
&\geq \Phi_{E\setminus E^{mono}}(\overrightarrow{p_i}) - 4\sum_{\{u,v\}\in E^{mono}} cost(uv)(\overrightarrow{p_i})_u(\overrightarrow{p_i})_v \\
&\geq \Phi(\overrightarrow{p_i}) - 4\frac{1}{40I}\Phi(\overrightarrow{p_i}) = \left(1 - \frac{1}{10I}\right)\Phi(\overrightarrow{p_i})
\end{aligned}$$

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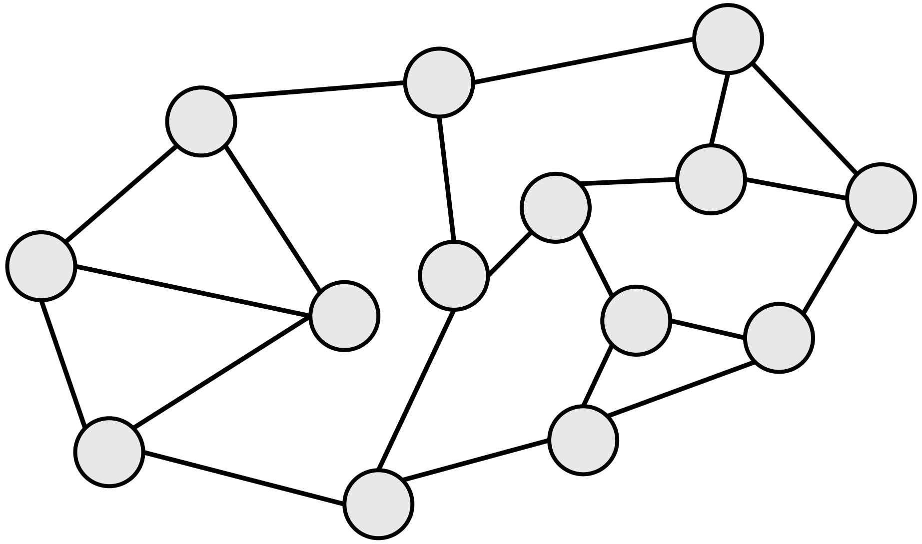

$\Delta = 3$

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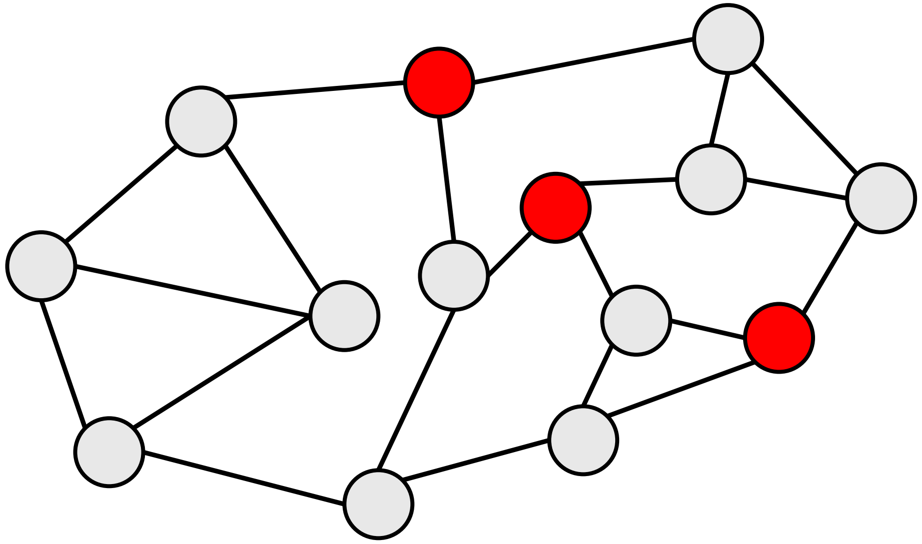

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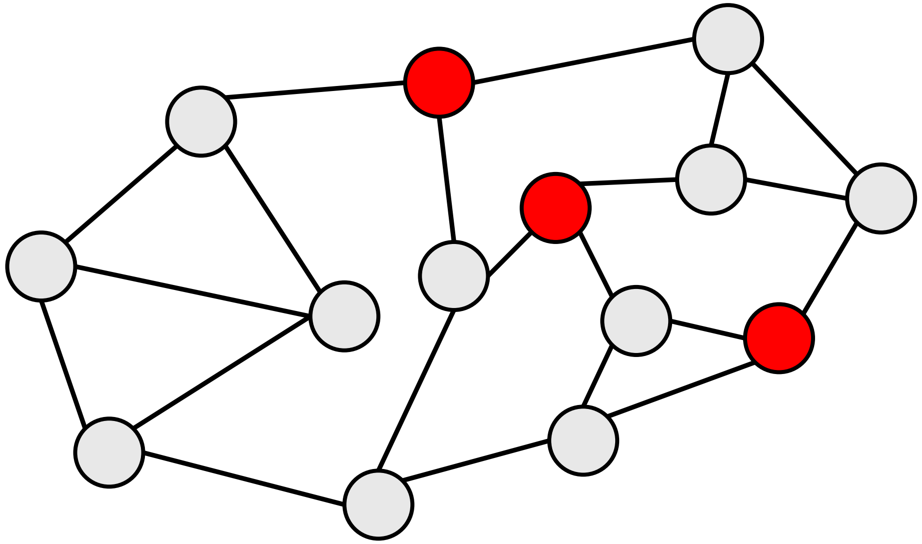

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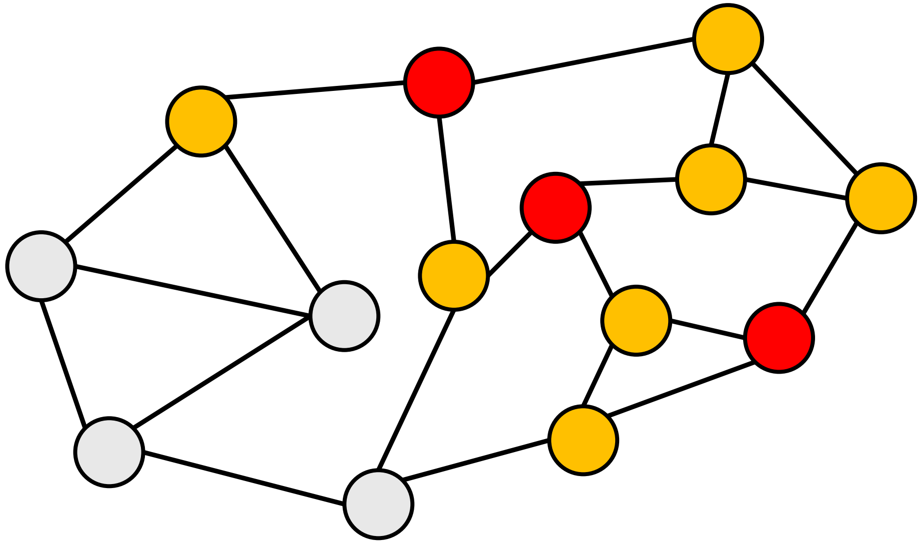

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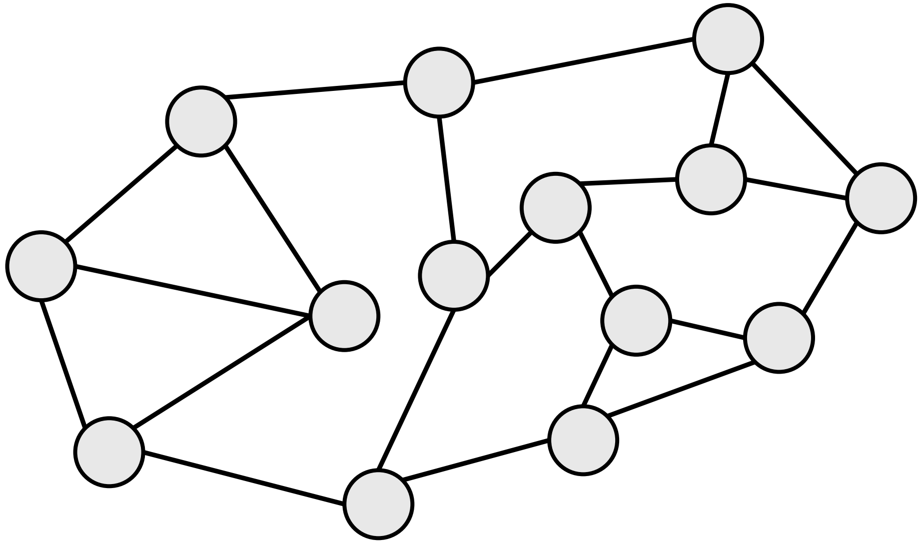

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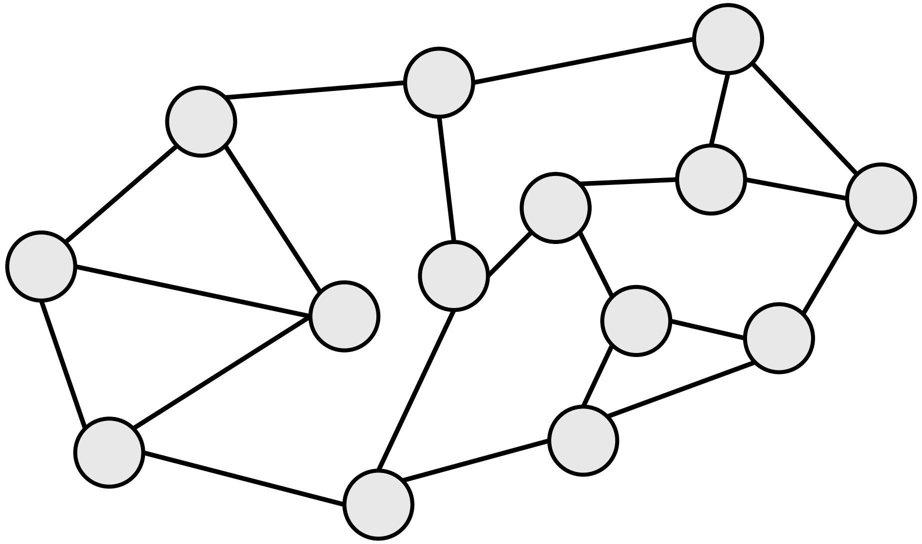

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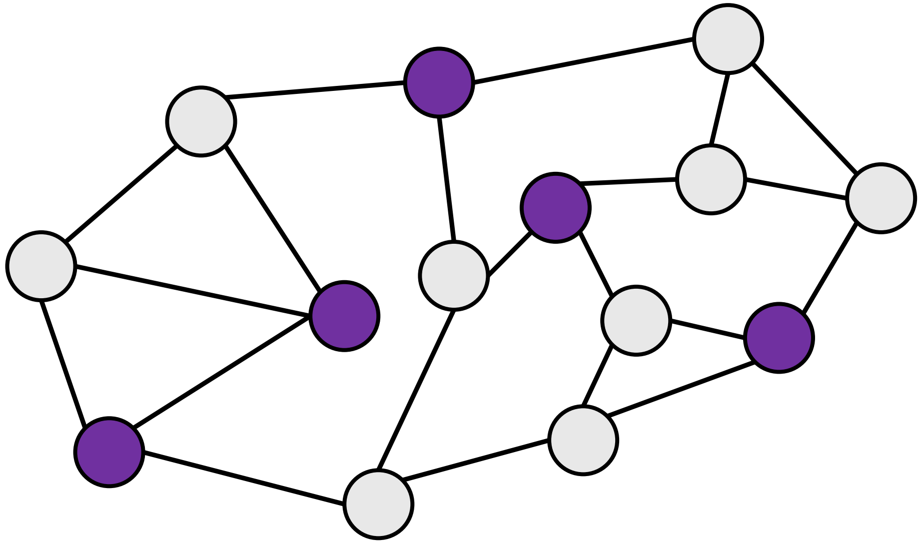

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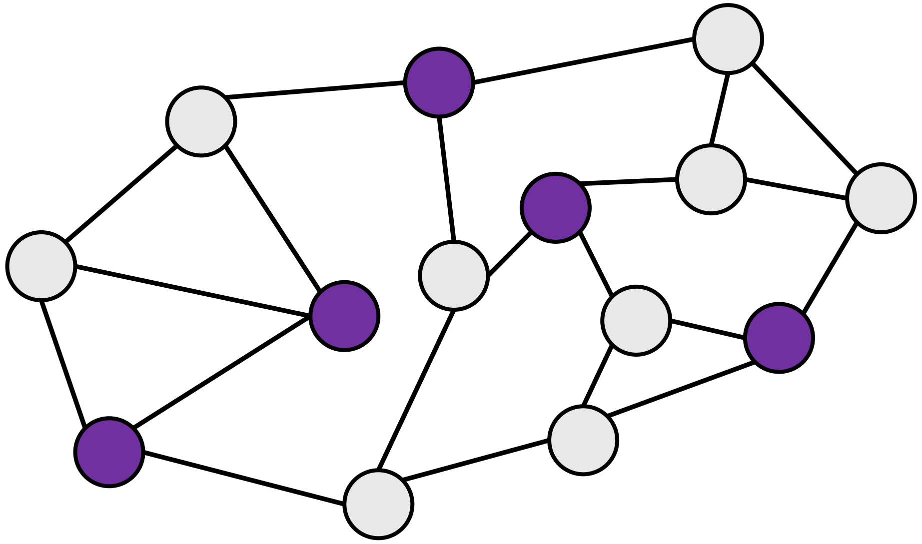

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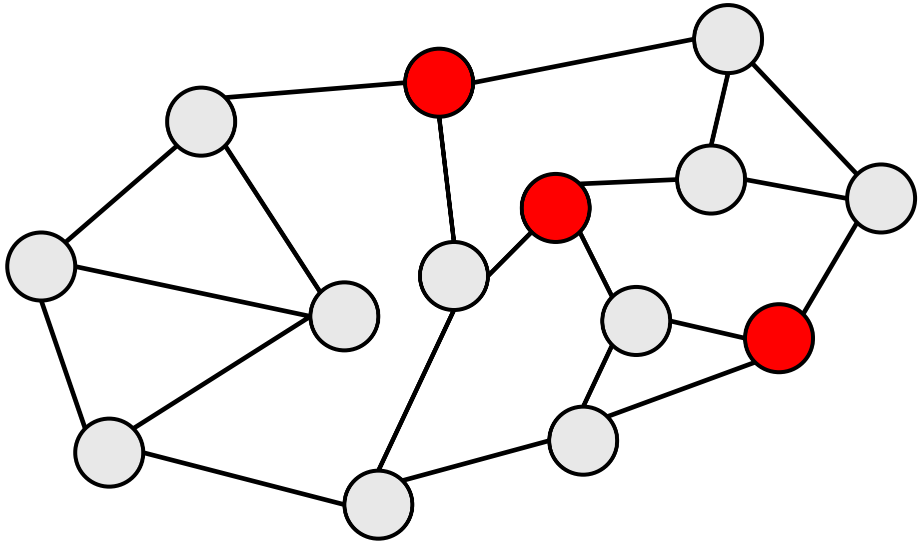

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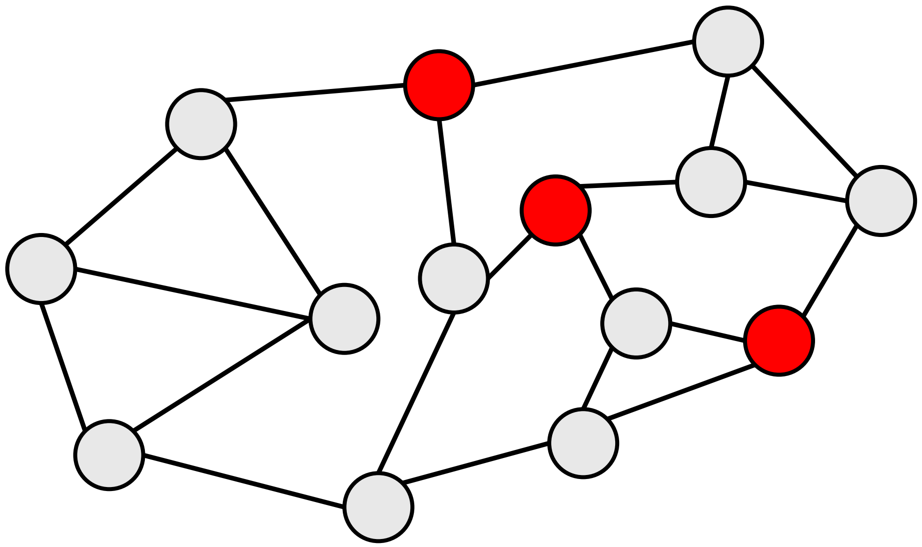

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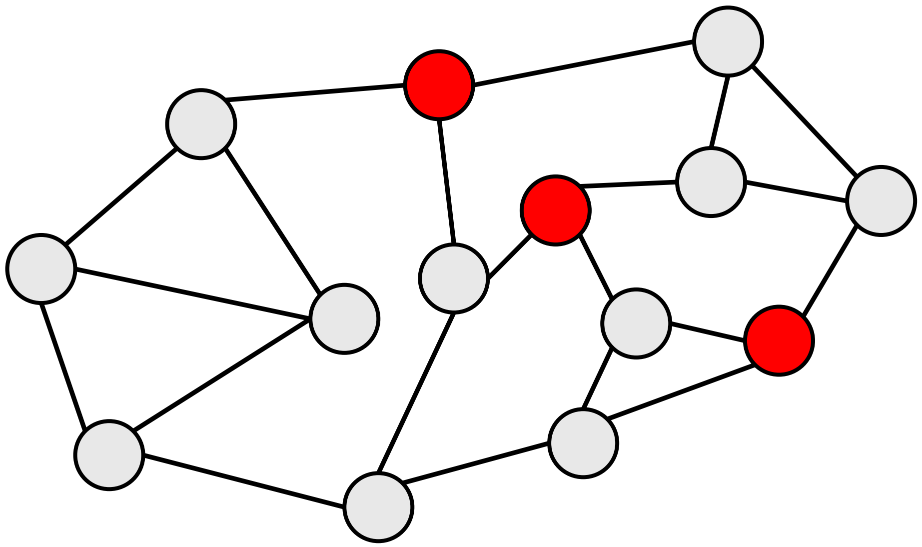

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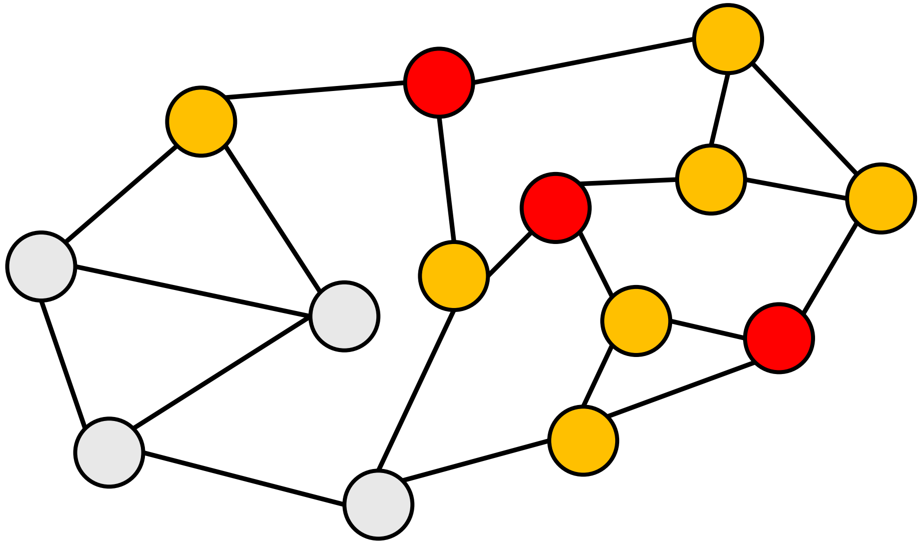

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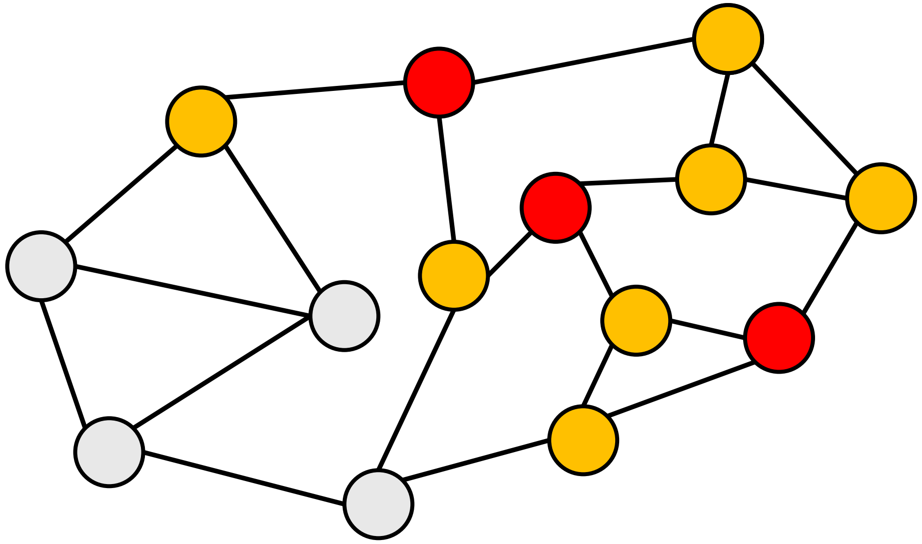

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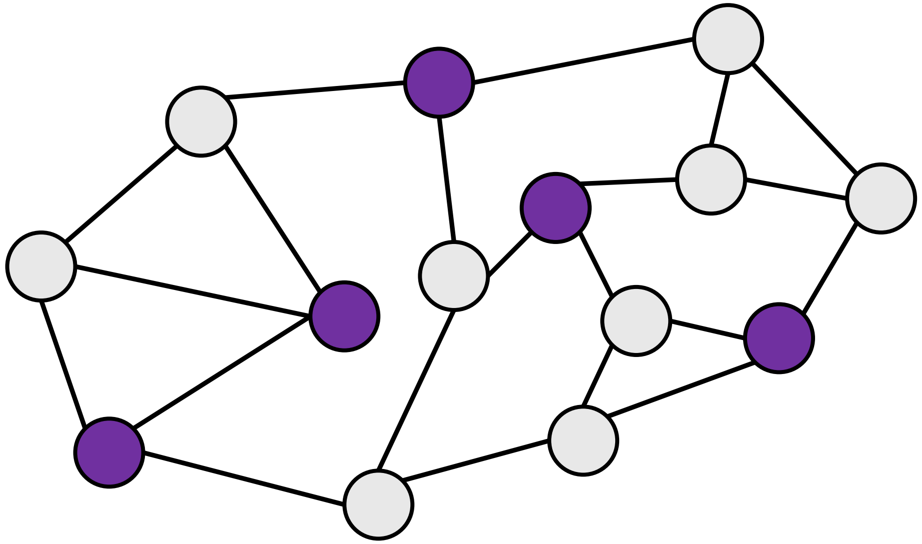

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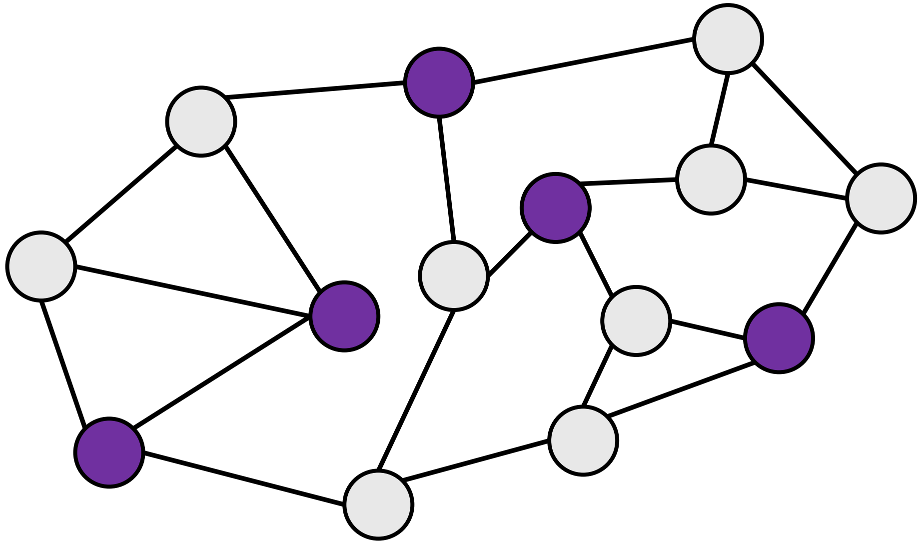

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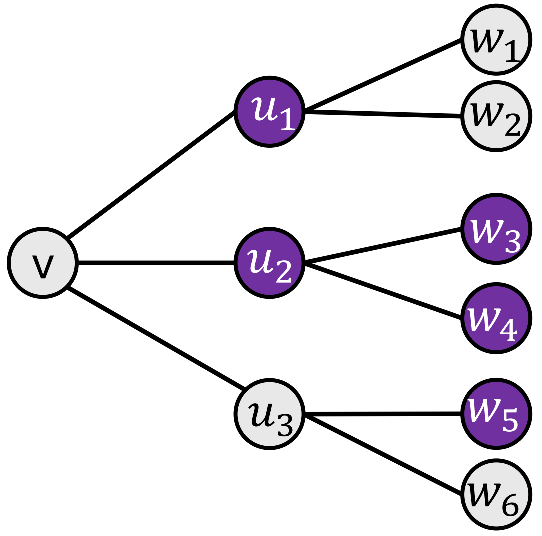

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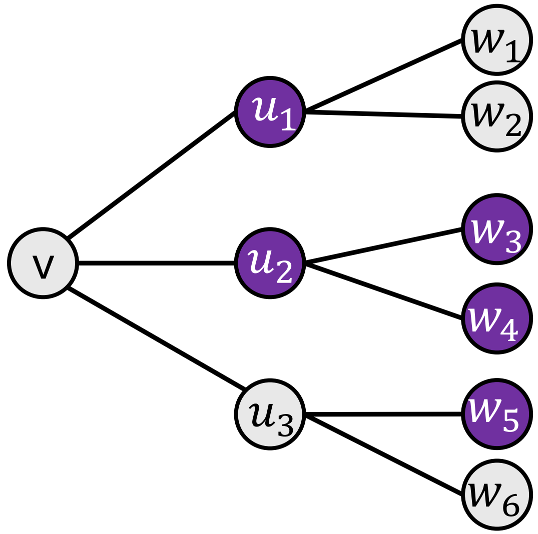

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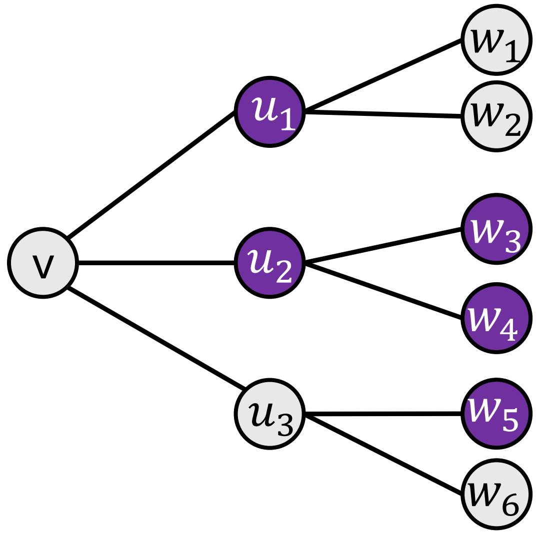

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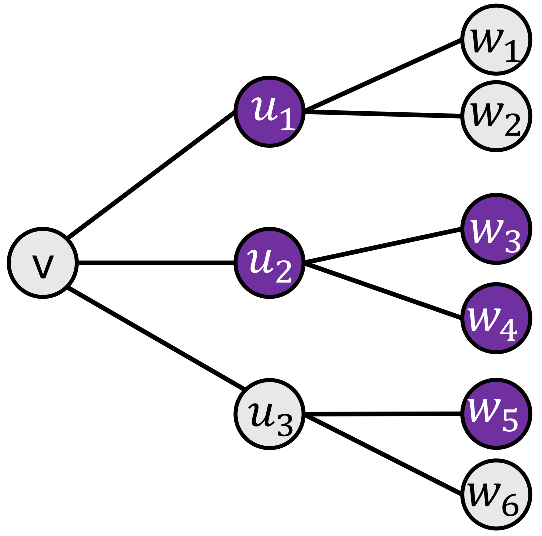

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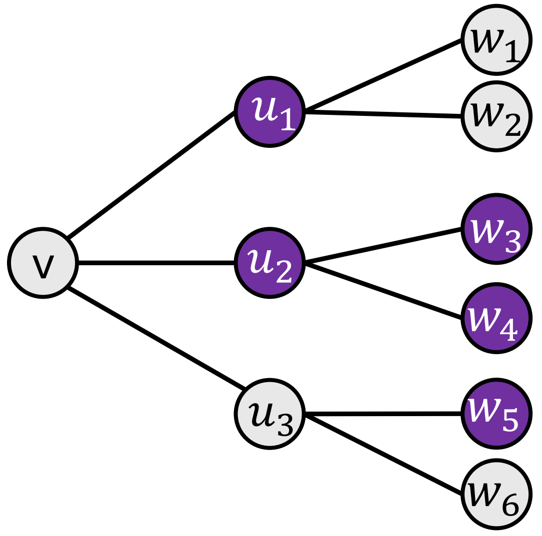

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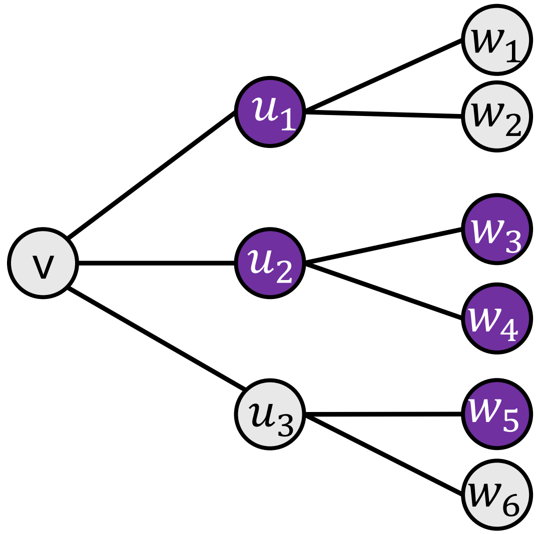

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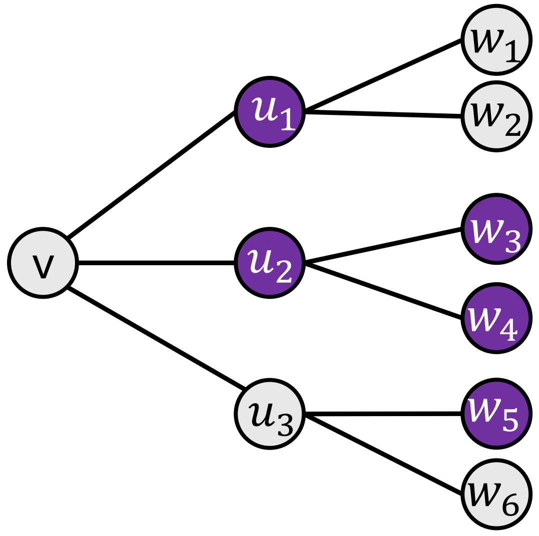

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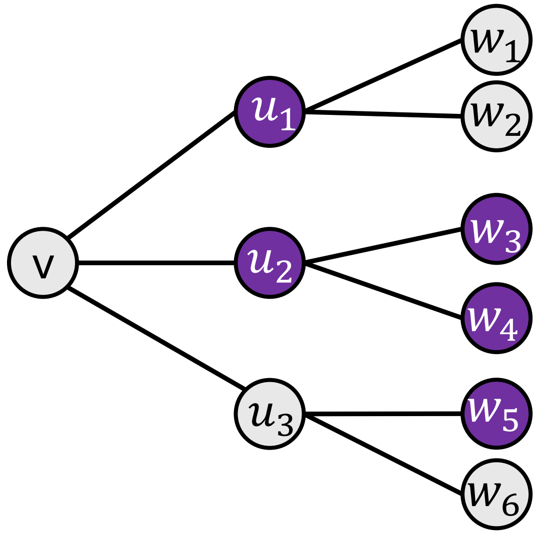

Claim 1:
Z_v is a pessimistic estimator of Y_v ($Z_v \leq Y_v$)

Claim follows from:

1.) $Z_v \leq 1$

2.) $Y_v = 0$ implies $Z_v \leq 0$

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Therefore,

$$E[Z_v] = \sum_{u \in N(v)} E[X_u] - \sum_{u \neq u' \in N(v)} E[X_u X_{u'}] - \sum_{u \in N(v)} \sum_{w \in N(u)} E[X_u X_w]$$
$$= \Delta \cdot \frac{1}{10\Delta} - \binom{\Delta}{2} \cdot \frac{1}{100\Delta^2} - \Delta^2 \cdot \frac{1}{100\Delta^2}$$
$$\geq \frac{1}{10} - \frac{1}{100} - \frac{1}{100}$$

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Claim 1 implies: $\sum_v Y_v \geq \sum_v Z_v$
Claim 2 implies: E[$\sum_v Z_v$] ≥ n/20
Therefore, the expected number of nodes neighboring **I** is at least n/20 ☺

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By the Local Rounding Theorem, this gives:

Corollary:
In $O(\log^2(\Delta) + \log^*(n))$ rounds, one can deterministically compute an independent set neighboring $\Omega(n)$ nodes.

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In fact, with slightly more work, one can show that each round of Luby's MIS algorithm can be derandomized in $O(\log^2(\Delta))$ rounds

MIS via Local Rounding

Theorem [FG**G**KR ’23] **:**
An MIS can be deterministically computed in $O(log\, n\, log^2\, \Delta)$ rounds.

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What if Δ is larger?

Technique #2: Intra-Cluster Rounding

Key Definition: Cluster Degree

Let C be a partition of the nodes into clusters.

The **cluster degree** of a node v is defined as

$$\deg_C(v) = \text{number of clusters containing a neighbor of } v.$$

The **maximum cluster degree** is

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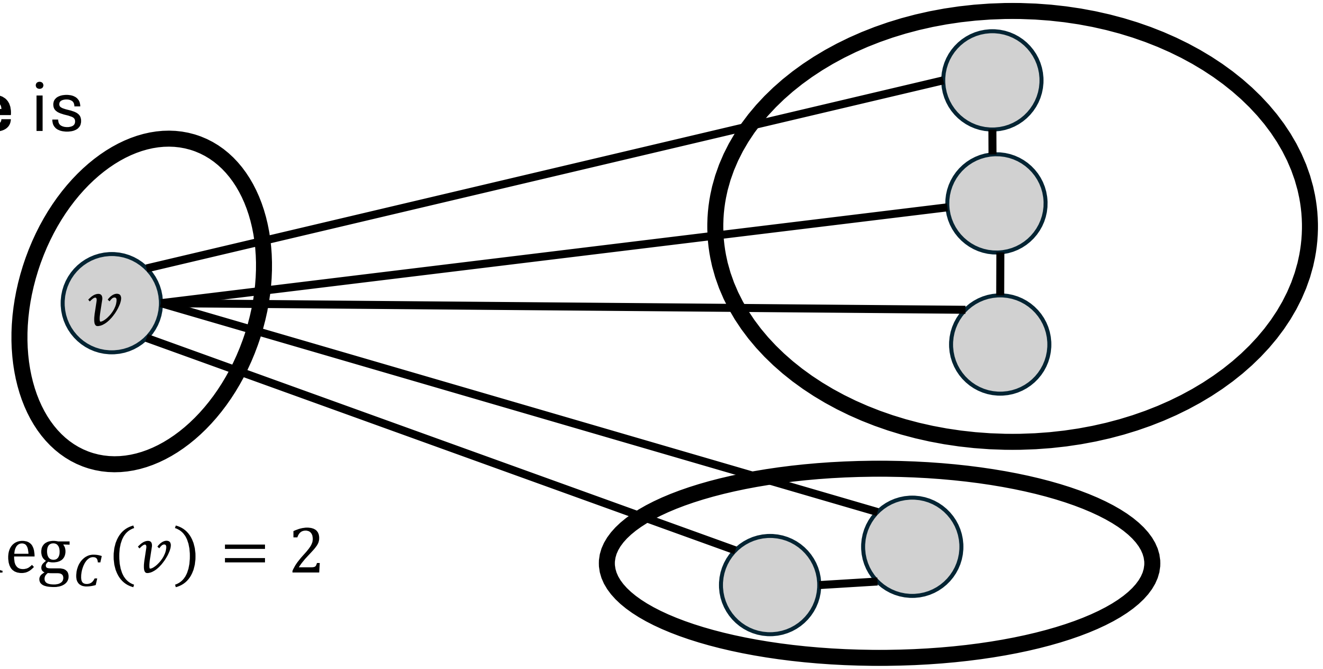

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Intra-Cluster Rounding

High-Level Idea: Round within clusters to increase integrality from $\approx \frac{1}{\Delta}$ to $\approx \frac{1}{\Delta_C}$

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Partial Intra-Cluster Rounding Theorem [Ghaffari, G '23] **:**
Let C be a partition of diameter D. In O(D) rounds, we can compute a $\frac{1}{10^9 \Delta_C \log(n)}$- integral vector such that if we treat the entries as sampling probabilities, we remove a constant fraction of the edges, in expectation.

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- Each cluster rounds independently – hence O(D) rounds.
- A "good" local solution is guaranteed by a probabilistic method argument.

Intra-Cluster + Local Rounding

Intra-Cluster + Local Rounding Theorem:
Given a partition C into clusters of diameter D, one can compute an MIS in $\widetilde{\boldsymbol{O}}(\log(n)\,(D + \log^2(\Delta_C)))$ rounds.

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Key Question:

What's the right trade-off between the diameter D and the max cluster degree Δ_C?

Clustering with Small Cluster Degree

Randomized Low-Diameter Decomposition Miller, Peng, Xu [SPAA'13] **:**
In $O(D)$ rounds, the MPX algorithm computes a partition with diameter D and cluster degree $n^{O(1/D)}$ (for $D \leq \log^{0.99}(n)$).

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<u>Observation</u>: Each cluster has diameter $O(D)$, w.h.p.

Technique #3: Derandomizing the randomized MPX algorithm

LDD with Small Cluster Degree

Derandomized MPX Decomposition Ghaffari, G [STOC'23] **:**
In $\widetilde{\boldsymbol{O}}(\log(n)\, D^2)$ rounds, one can compute a partition with diameter D and maximum cluster degree $n^{O(1/D)}$ (assuming $D \leq \log^{0.99}(n)$).

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High-Level Idea:

- Each node has D random coin tosses that define its head start.
- Derandomize coin #1 of all nodes, then coin #2, and so on -- for D rounds.
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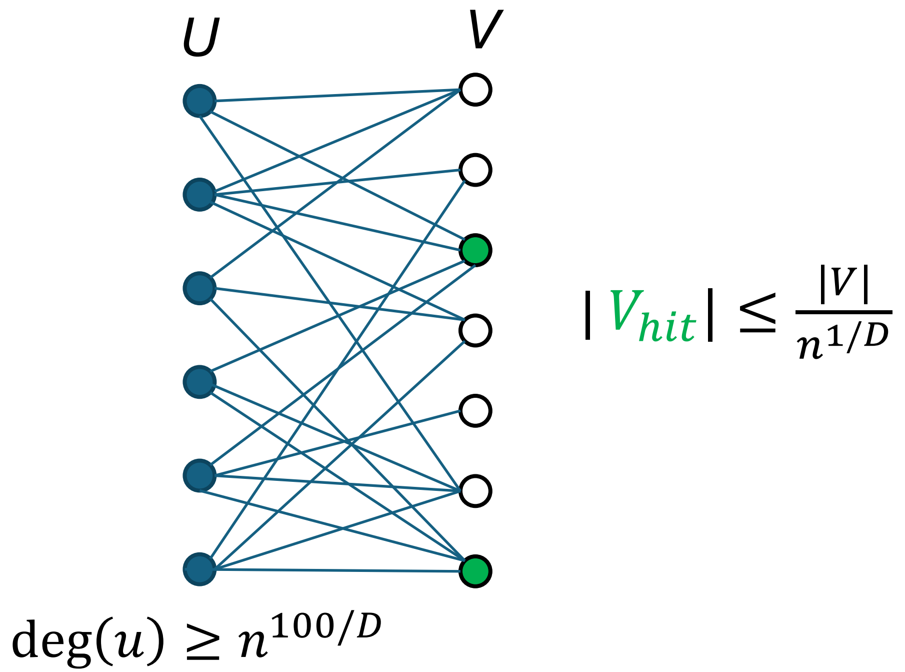
Corollary Ghaffari, G [STOC'23] **:**
An MIS can be computed in $\tilde{O}(\log^2 n)$ rounds.

Technique #4: Deterministic Hitting Set

Deterministic Hitting Set

Deterministic Hitting Set Ghaffari, G [STOC'23] **:**
Consider a bipartite graph with bipartition $U \sqcup V$, where each $u \in U$ has degree $\deg(u) \geq n^{100/D}$.
In $\widetilde{\boldsymbol{O}}(\log(n))$ rounds, one can compute a subset $V_{hit} \subseteq V$ such that:

1. $|V_{hit}| \leq \frac{|V|}{n^{1/D}}$
2. $N(u) \cap V_{hit} \neq \emptyset$ for every $u \in U$.



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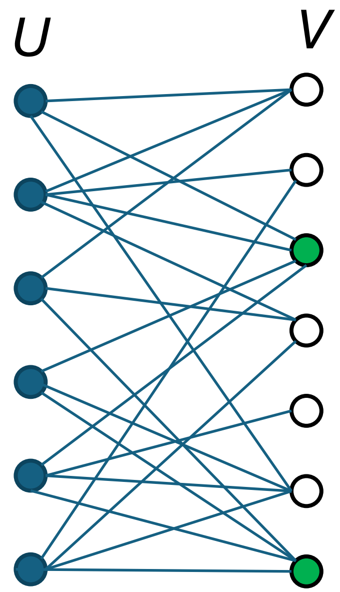

$\deg(u) \geq n^{100/D}$

$|V_{hit}| \leq \frac{|V|}{n^{1/D}}$

Simple 0-round randomized algorithm!

Open Problems

Thanks!

- Deterministic MIS (or Maximal Matching, $(\Delta + 1)$-coloring) in o($\log^{5/3} n$) or o($log\, n\, log^2\, \Delta$) rounds

- Derandomization Gap in Distributed Graph Algorithms (for locally checkable problems)
  - Det/Rand =$\tilde{O}(\log^2 n)$ G., Grunau FOCS'24
  - Det/Rand = $\tilde{\Omega}(\log n)$ Brandt et al. STOC'16; Chang, Pettie, Kopelowitz FOCS'17; G., Su SODA'17
  - ? Close the gap?

- Use Local Rounding in other models
  - Parallel derandomization, towards work-efficiency G., Grunau '25
  - ? Dynamic, ...?